Schubert calculus and cohomology of Lie groups

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SUSTECH

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The contents of the talk

E. Cartan(1929): Determine the cohomology $H^*(G;\mathbb{F})$ (with $\mathbb{F} = \mathbb{R},\ \mathbb{F}_p$ or $\mathbb{Z}$) of compact Lie groups G.

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Backgrounds of the problem:

- Cartan has completed his classification on the Lie groups and symmetric spaces (1928)
- Lefschetz has secured the foundation of the homology and cohomology theory for the cell-complexes (1927).

The contents of the talk

1. Classical methods and the remaining problem
2. Schubert calculus
3. The integral cohomology of Lie groups (Duan, Zhao)

Classical methods and the remaining problem

Theorem 1

Every compact, connected and finite dimensional Lie group G has the canonical form:

$$G = (G_1 \times \cdots \times G_k \times T^r)/K,$$

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3. K *is a finite subgroup of the center of* $G_1 \times \cdots \times G_k \times T^r$.

Therefore, we shall adopt the convention in this talk:

"the Lie groups G under our consideration are the 1-connected and simple ones listed above."

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Up to 1936 Brauer and Pontryagin have determined the algebra $H^*(G;\mathbb{R})$ for the classical groups

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Based on the cell decompositions on these groups Pontryagin (1938) obtained that

Theorem 2

The Poincare polynomials of the classical groups are

- $P_t(Spin(2n+1)) = (1-t^3)(1-t^7)\cdots(1-t^{4n-1})$;
- $P_t(Spin(2n)) = (1-t^3)(1-t^7)\cdots(1-t^{4n-5})(1-t^{2n-1})$;
- $P_t(SU(n)) = (1-t^3)(1-t^5)\cdots(1-t^{2n-1})$;
- $P_t(Sp(n)) = (1-t^3)(1-t^7)\cdots(1-t^{4n-1})$.

Classical methods and the remaining problem

A remarkable change began with Hopf in 1941, who studied the following problem:

Problem

Classify those algebras $\alpha : A \otimes A \to A$ *over the reals that can be furnished with a co-product* $\beta : A \to A \otimes A$.

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The group of multiplication $\mu : G \times G \to G$ on G induces a co-product

$$\beta = \mu^* : H^*(G;\mathbb{R}) \to H^*(G \times G;\mathbb{R}) = H^*(G;\mathbb{R}) \otimes H^*(G;\mathbb{R})$$

that furnishes the cohomology $H^*(G;\mathbb{R})$ with the structure of an **Hopf algebra**.

Classical methods and the remaining problem

In term of the co-product $\beta : A \to A \otimes A$ Hopf introduced the **set of the primative elements** in the algebra A

$$P(A) = \{a \in A \mid \beta(a) = a \otimes 1 \oplus 1 \otimes a\}.$$

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Since this is a real vector space over R one can take a basis

$$x_1, \cdots, x_n; y_1, \cdots, y_m$$

for $P(A)$ with $deg(x_i) =$ even and $deg(y_i) =$ odd.

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Theorem 3

$$A = \mathbb{R}[x_1, \cdots, x_n] \otimes \wedge_{\mathbb{R}}(y_1, \cdots, y_m)$$

Classical methods and the remaining problem

Corollary

If G is a finite dimensional Lie group, $H^*(G;\mathbb{R}) = \wedge_{\mathbb{R}}(y_1, \cdots, y_m)$.

Corollary

Conjecture of Cartan (1936):$H^*(G;\mathbb{R}) = H^*(S^{2n_1-1} \times \cdots \times S^{2n_k-1};\mathbb{R})$

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Corollary

Yan (1949): Let G be an exceptional Lie group. Then

1. $H^*(G_2;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_{11})$
2. $H^*(F_4;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_{11}, y_{15}, y_{23})$
3. $H^*(E_6;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_9, y_{11}, y_{15}, y_{17}, y_{23})$
4. $H^*(E_7;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_{11}, y_{15}, y_{19}, y_{23}, y_{27}, y_{35})$
5. $H^*(E_8;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_{15}, y_{23}, y_{27}, y_{35}, y_{39}, y_{47}, y_{59})$

Classical methods and the remaining problem

Borel (1953) initiated the work to compute the cohomology $H^*(G;\mathbb{F}_p)$ of Lie groups G with coefficients in a finite field $\mathbb{F}_p$.

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He began with the following classifying result:

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Let A be a finitely generated Hopf over a finite field $\mathbb{F}_p$. Then

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Let A be a finitely generated Hopf over a finite field $\mathbb{F}_p$. Then

$$A = B(x_1) \otimes \cdots \otimes B(x_n)$$

where each $B(x_i)$ is one of **the monogenic Hopf algebra** *over $\mathbb{F}_p$:*

$B(x_i)$	$\deg(x_i)$ *odd*	$\deg(x_i)$ *even*
$p \neq 2$	$\Lambda_{\mathbb{F}_p}(x_i)$	$\mathbb{F}_p(x_i)/(x_i^{p^r})$
$p = 2$	$\mathbb{F}_2(x_i)/(x_i^{2^r})$	$\mathbb{F}_2(x_i)/(x_i^{2^r})$

Classical methods and the remaining problem

Based on Borel's classification on the Hopf algebra, the cohomologies $H^*(G;\mathbb{F}_p)$ have been calculated case by case:

- Borel (1953) computed $H^*(G_2;\mathbb{F}_2), H^*(F_4;\mathbb{F}_2)$;
- Araki (1960) computed $H^*(F_4;\mathbb{F}_3)$;
- Toda, Kono, Mimura, Shimada (1973,75,76) obtained $H^*(E_i;\mathbb{F}_2)$, $i = 6, 7, 8$;
- Kono, Mimura (1975, 1977) obtained $H^*(E_i;\mathbb{F}_3)$, $i = 6, 7, 8$;
- Kono (1977) obtained $H^*(E_8;\mathbb{F}_5)$

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Example

The cohomology algebra $H^*(E_8;\mathbb{F}_p)$ of the exceptional Lie group E_8 is

- *If* $p=2: \frac{\mathbb{F}_2[\alpha_3,\alpha_5,\alpha_9,\alpha_{15}]}{\langle\alpha_3^{16},\alpha_5^{8},\alpha_9^{4},\alpha_{15}^{4}\rangle}\otimes\Lambda_{\mathbb{F}_2}(\alpha_{17},\alpha_{23},\alpha_{27},\alpha_{29})$
- *If* $p=3:\mathbb{F}_3[x_8,x_{20}]/\langle x_8^3,x_{20}^3\rangle\otimes\Lambda_{\mathbb{F}_3}(\zeta_3,\zeta_7,\zeta_{15},\zeta_{19},\zeta_{27},\zeta_{35},\zeta_{39},\zeta_{47})$;
- $p=5:\mathbb{F}_5[x_{12}]/\langle x_{12}^5\rangle\otimes\Lambda_{\mathbb{F}_5}(\zeta_3,\zeta_{11},\zeta_{15},\zeta_{23},\zeta_{27},\zeta_{35},\zeta_{39},\zeta_{47})$.

Classical methods and the remaining problem

The problem we shall study is:

Problem

Find a **unified construction** *of the integral cohomology ring* $H^*(G)$ *of compact Lie groups* G*, in particular, of* $G = G_2, F_4, E_6, E_7, E_8$.

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1. The integral cohomology $H^*(G)$ for $G = SU(n), Sp(n), Spin(n)$ has been determined by Borel (1952) and Pittie (1991).

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1. The integral cohomology $H^*(G)$ for $G = SU(n), Sp(n), Spin(n)$ has been determined by Borel (1952) and Pittie (1991).
2. the integral cohomology $H^*(G)$ fails to be an Hopf ring:

$$\beta = \mu^* : H^*(G) \to H^*(G \times G) \cong H^*(G) \otimes H^*(G) \oplus Ext(H^*(G), H^*(G))$$

1) the isomorphism $\cong$ given by the Kunneth formula is additive, but is not multiplicative;
2) the appearance of the summand $Ext(H^*(G), H^*(G))$

Schubert calculus

Schubert calculus is the intersection theory founded by Poncellet, Charles, Schubert, $\cdots$, in the 19th century, together with its applications to enumerative geometry:

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Example

- *Given 8 quadrics in the space P^3, find the number of conics tangent to all 8. (4,407,296)*
- *Given 9 quadrics in the space P^3, find the number of quadrics tangent to all 9. (666,841,088)*
- *Given 12 quadrics in the space P^3, find the number of twisted cubic space curves tangent to all 12. (5,819,539,783,680).*

References:

- H. Schubert, Kalkul der abzahlenden Geometrie, 1879.

Schubert calculus

Schubert based his calculation on **the principle of continouity** due to Poncelet (1814), which was bitterly attacked by Cauchy even before the publication of Poncelet's treatise (1822).

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Cauchy's judgement led to prejudice against Schubert's works $\cdots$

Leidheuser introduces the intersection multiplicity into the debate.
This brings in Ecc. Francesco Severi, Rome.

References

- S. Kleiman, Rigorous foundation of Schubert's enumerative calculus, 1976.
- N. Schappacher, The unreasonable resilience of calculations, 2008.

Schubert calculus

Hilbert's 15th problem: Rigorous foundation of Schubert's enumerative calculus.

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A. Weil, 1948: "The classical Schubert calculus amounts to the determination of cohomology rings $H^*(G/P)$ of flag manifolds G/P" where G is a Lie group and $P \subset G$ is a parabolic subgroup.

References

- van der Waerden, Topologische Begrundung des Kalkuls der abzahlenden Geometrie, 1929.
- A. Weil, Foundation of algebraic geometry, 1948.

Schubert calculus

Chevalley announced in 1958 that the Schubert varieties $\{X_w \subset G/P \mid w \in W\}$ on a flag manifold G/P provide a cell decomposition

$$G/P = \cup_{w \in W} X_w, \quad dim_R X_w = 2l(w), \ W = W(G)/W(P).$$

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Consequently, one has **the basis theorem of Schubert calculus:**

Theorem 5

On a flag manifold G/P the set $\{s_w, w \in W\}$ of Schubert classes forms an additive basis of $H^*(G/P)$.

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and the **fundamental problem of Schubert calculus:**

Problem

Determine the structure constants $c_{u,v}^w \in Z$ required to expand the product of two arbitrary Schubert classes

$$s_u \cup s_v = \sum c_{u,v}^w \cdot s_w.$$

Schubert calculus

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Example

The table of structure constants of the variety of complete conics on P^3 (from Schubert's book in 1879)

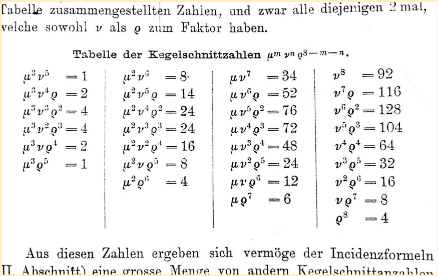

Tabelle zusammengestellten Zahlen, und zwar alle diejenigen 2 mal, welche sowohl ν als ϱ zum Faktor haben.

Tabelle der Kegelschnittzahlen $\mu^m \nu^n \varrho^{8-m-n}$.

$\mu^3\nu^5 = 1$	$\mu^2\nu^6 = 8$	$\mu\nu^7 = 34$	$\nu^8 = 92$
$\mu^3\nu^4\varrho = 2$	$\mu^2\nu^5\varrho = 14$	$\mu\nu^6\varrho = 52$	$\nu^7\varrho = 116$
$\mu^3\nu^3\varrho^2 = 4$	$\mu^2\nu^4\varrho^2 = 24$	$\mu\nu^5\varrho^2 = 76$	$\nu^6\varrho^2 = 128$
$\mu^3\nu^2\varrho^3 = 4$	$\mu^2\nu^3\varrho^3 = 24$	$\mu\nu^4\varrho^3 = 72$	$\nu^5\varrho^3 = 104$
$\mu^3\nu\varrho^4 = 2$	$\mu^2\nu^2\varrho^4 = 16$	$\mu\nu^3\varrho^4 = 48$	$\nu^4\varrho^4 = 64$
$\mu^3\varrho^5 = 1$	$\mu^2\nu\varrho^5 = 8$	$\mu\nu^2\varrho^5 = 24$	$\nu^3\varrho^5 = 32$
	$\mu^2\varrho^6 = 4$	$\mu\nu\varrho^6 = 12$	$\nu^2\varrho^6 = 16$
		$\mu\varrho^7 = 6$	$\nu\varrho^7 = 8$
			$\varrho^8 = 4$

Aus diesen Zahlen ergeben sich vermöge der Incidenzformeln (II. Abschnitt) eine grosse Menge von andern Kegelschnittanzahlen

Schubert calculus

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The table of structure constants of the variety of complete quadrics on P^3 (from Schubert's book in 1879)

Elementarzahlen der F_2 heissen, sind in der folgenden Tabelle zusammengestellt.

Tabelle der Anzahlen $\mu^m \nu^n \varrho^{9-m-n}$ **für die Fläche zweiten Grades.**

$\mu^9 = \varrho^9 = 1$	$\nu^2\mu^7 = \nu^2\varrho^7 = 4$	$\nu^4\mu^3\varrho^2 = \nu^4\mu^2\varrho^3 = 112$
$\mu^8\varrho = \mu\varrho^8 = 3$	$\nu^2\mu^6\varrho = \nu^2\mu\varrho^6 = 12$	$\nu^5\mu^4 = \nu^5\varrho^4 = 32$
$\mu^7\varrho^2 = \mu^2\varrho^7 = 9$	$\nu^2\mu^5\varrho^2 = \nu^2\mu^2\varrho^5 = 36$	$\nu^5\mu^3\varrho = \nu^5\mu\varrho^3 = 80$
$\mu^6\varrho^3 = \mu^3\varrho^6 = 17$	$\nu^2\mu^4\varrho^3 = \nu^2\mu^3\varrho^4 = 68$	$\nu^5\mu^2\varrho^2 = \nu^5\mu^2\varrho^2 = 128$
$\mu^5\varrho^4 = \mu^4\varrho^5 = 21$	$\nu^3\mu^6 = \nu^2\varrho^6 = 8$	$\nu^6\mu^3 = \nu^6\varrho^3 = 56$
$\nu\mu^8 = \nu\varrho^8 = 2$	$\nu^3\mu^5\varrho = \nu^3\mu\varrho^5 = 24$	$\nu^6\mu^2\varrho = \nu^6\mu\varrho^2 = 104$
$\nu\mu^7\varrho = \nu\mu\varrho^7 = 6$	$\nu^3\mu^4\varrho^2 = \nu^3\mu^2\varrho^4 = 72$	$\nu^7\mu^2 = \nu^7\varrho^2 = 80$
$\nu\mu^6\varrho^2 = \nu\mu^2\varrho^6 = 18$	$\nu^3\mu^3\varrho^3 = \nu^3\mu^3\varrho^3 = 104$	$\nu^7\mu\varrho = \nu^7\mu\varrho = 104$
$\nu\mu^5\varrho^3 = \nu\mu^3\varrho^5 = 34$	$\nu^4\mu^5 = \nu^4\varrho^5 = 16$	$\nu^8\mu = \nu^8\varrho = 92$
$\nu\mu^4\varrho^4 = \nu\mu^4\varrho^4 = 42$	$\nu^4\mu^4\varrho = \nu^4\mu\varrho^4 = 48$	$\nu^9 = \nu^9 = 92$

Hiernach kann man nun auch alle diejenigen neunfachen Bedingungen berechnen, die einen Faktor enthalten, den man als

"The fundamental problem which occupies Schubert is to express the product of two of these symbols in terms of others linearly. He succeeds in part."

J. Coolidge, A History of Geometrical Methods, Oxford, 1940.

Schubert calculus

The solution to the "fundamental problem of Schubert calculus" has been obtained by Duan in the works

Multiplicative rule of Schubert classes, Invent. Math. 2005,2009.

Theorem 6

If $u, v \in W$ with $l(w) = l(u) + l(v)$ then

$$c_{u,v}^{w} = T_{A_w}\Big[\Big(\sum_{\substack{|L|=l(u)\\ \sigma_L=u}} x_L\Big)\Big(\sum_{\substack{|K|=l(v)\\ \sigma_K=v}} x_K\Big)\Big]$$

This formula

- express the structure constant $c_{u,v}^{w}$ as a polynomial in the Cartan numbers of the Lie group G;

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$$c_{u,v}^{w} = T_{A_w}[(\sum_{\substack{|L|=l(u)\\ \sigma_L=u}} x_L)(\sum_{\substack{|K|=l(v)\\ \sigma_K=v}} x_K)]$$

This formula

- express the structure constant $c_{u,v}^{w}$ as a polynomial in the Cartan numbers of the Lie group G;
- applies uniformly to all flag manifold G/P.

Schubert calculus

Based on the formula, a computer package entitled "The Chow ring of flag varieties" has been composed in the following works

- Duan+Zhao, The Chow ring of a generalized Grassmannian, Found. Math. comput. 2010.
- Duan+Zhao, Schubert presentations of complete flag manifolds G/T, LMS.J.Comput.Math. 2015.

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whose function can be described as follows:

Algorithm

Input: *The Cartan matrix* $A = (a_{ij})_{n\times n}$ *of the Lie group* G*, and a subset* $I \subseteq \{1, 2, \cdots, n\}$ *(to specify the parabolic subgroup* $P \subset G$*)*

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Output:

- *A minimal set of Schubert classes on* G/P *that generates the cohomology* $H^*(G/P)$ *multiplicatively;*
- *A presentation of the ring* $H^*(G/P)$ *in these generators.*

Schubert calculus

Example

$$H^*(G_2/T) = \mathbb{Z}[\omega_1, \omega_2, y_3]/\langle \rho_2, r_3, r_6 \rangle, \text{ where} \tag{5.1}$$
$$\rho_2 = 3\omega_1^2 - 3\omega_1\omega_2 + \omega_2^2;$$
$$r_3 = 2y_3 - \omega_1^3;$$
$$r_6 = y_3^2.$$

$$H^*(F_4/T) = \mathbb{Z}[\omega_1, \omega_2, \omega_3, \omega_4, y_3, y_4]/\langle \rho_2, \rho_4, r_3, r_6, r_8, r_{12} \rangle \text{ where} \tag{5.2}$$
$$\rho_2 = c_2 - 4\omega_1^2;$$
$$\rho_4 = 3y_4 + 2\omega_1 y_3 - c_4;$$
$$r_3 = 2y_3 - \omega_1^3;$$
$$r_6 = y_3^2 + 2c_6 - 3\omega_1^2 y_4;$$
$$r_8 = 3y_4^2 - \omega_1^2 c_6;$$
$$r_{12} = y_4^3 - c_6^2.$$

$$H^*(E_6/T) = \mathbb{Z}[\omega_1, \dots, \omega_6, y_3, y_4]/\langle \rho_2, \rho_3, \rho_4, \rho_5, r_6, r_8, r_9, r_{12} \rangle, \text{ where} \tag{5.3}$$
$$\rho_2 = 4\omega_2^2 - c_2;$$
$$\rho_3 = 2y_3 + 2\omega_2^3 - c_3;$$
$$\rho_4 = 3y_4 + \omega_2^4 - c_4;$$
$$\rho_5 = 2\omega_2^2 y_3 - \omega_2 c_4 + c_5;$$
$$r_6 = y_3^2 - \omega_2 c_5 + 2c_6;$$
$$r_8 = 3y_4^2 - 2c_5 y_3 - \omega_2^2 c_6 + \omega_2^3 c_5;$$
$$r_9 = 2y_3 c_6 - \omega_2^3 c_6;$$
$$r_{12} = y_4^3 - c_6^2.$$

Schubert calculus

For a Lie group G with a maximal torus T, let $\{\omega_1, \cdots, \omega_n \in H^2(G/T)\}$ be the set of fundamental dominant weights of G.

Theorem 7

The ring $H^*(G/T)$ has the presentation

$$H^*(G/T) = \mathbb{Z}[\omega_1, \cdots, \omega_n, y_1, \cdots, y_m] / \langle e_i, f_j, g_j \rangle_{1 \leq i \leq n-m; 1 \leq j \leq m}$$

where the polynomial relations e_i, f_j, g_j have the following properties:

1. *for each $1 \leq i \leq n-m$,* $e_i \in \langle \omega_1, \cdots, \omega_n \rangle$;
2. *for each $1 \leq j \leq m$*

$$f_j = p_j \cdot y_j + \alpha_j, \quad g_j = y_j^{k_j} + \beta_j,$$

with $p_j \in \{2, 3, 5\}$ and α_j, $\beta_j \in \langle \omega_1, \cdots, \omega_n \rangle$. □

Schubert calculus

- Borel has shown in 1953 that, over the field $\mathbb{R}$ of reals

$$H^*(G/T;\mathbb{R}) = \mathbb{R}[\omega_1, \cdots, \omega_n]/\mathbb{R}[\omega_1, \cdots, \omega_n]^{W+}.$$

But it fails if we replace $\mathbb{R}$ by the ring $\mathbb{Z}$ or by a finite field;

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But it fails if we replace $\mathbb{R}$ by the ring $\mathbb{Z}$ or by a finite field;

- We shall see in the coming section that,
  1. the set $\{y_1, \cdots, y_m\}$ of Schubert classes on G/T, and
  2. the polynomials e_i, α_j, β_j in the Schubert classes $\omega_1, \cdots, \omega_n, y_1, \cdots, y_m$ on G/T

  are just what requested to construct the integral cohomology rings $H^*(G)$ **uniformly for all Lie group G.**

Cohomology of Lie groups

For a Lie group G with a maximal torus T consider the fibration

$$T \hookrightarrow G \xrightarrow{\pi} G/T.$$

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Our construction uses the Leray Serre spectral sequence $\{E_r^{*,*}(G), d_r\}$ in which one has

- $E_2^{*,*}(G) = H^*(G/T) \otimes H^*(T)$;
- $d_2(x \otimes t) = x \cup \tau(t) \otimes 1,\ x \in H^*(G/T),\ t \in H^1(T).$

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Added to this we have three crucial facts:

- the factor ring $H^*(G/T)$ of $E_2^{*,*}(G)$ has been made explicit in Duan+Zhao, LMS.J.Comput.Math. 2015;

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Added to this we have three crucial facts:

- the factor ring $H^*(G/T)$ of $E_2^{*,*}(G)$ has been made explicit in Duan+Zhao, LMS.J.Comput.Math. 2015;
- the Borel transgression $\tau : H^1(T) \to H^2(G/T)$ has been decided by Duan, in Homology, Homotopy Appl. 2018;
- $E_3^{*,*}(G) = H^*(G)$ (conjectured by Marlin in 1989).

Combining Schubert presentation of the ring $H^*(G/T)$ with the term $\{E_2^{*,*}(G), d_2\}$ we construct below three types of explicit elements

$$x_i,\ \varrho_k,\ \mathcal{C}_I \in H^*(G)$$

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Type I. In view of the quotient map $\pi : G \to G/T$, the set $\{y_1, \cdots, y_m\}$ of special Schubert classes on G/T yields directly the integral cohomology classes of the group G

$$x_{\deg y_i} := \pi^*(y_i) \in H^*(G), 1 \leq i \leq m$$

.

Type II. In view of the ready made maps

$$\langle \omega_1, \cdots, \omega_n \rangle \xrightarrow{\varphi} E_2^{2k,1}(G), \quad E_3^{2k,1}(G) \xrightarrow{\kappa} H^{2k+1}(G)$$

$$\varphi(g_1\omega_1 + \cdots + g_n\omega_n) = g_1 \otimes t_1 + \cdots + g_n \otimes t_n$$

and the commutative diagrams

$$\begin{array}{ccccc} & & E_3^{*,1}(G) & \xrightarrow{\kappa} & H^*(G) \\ & \widetilde{\varphi} \nearrow & \downarrow & & \\ \langle \omega_1, \cdots, \omega_n \rangle & \xrightarrow{\varphi} & E_2^{*,1}(G) & & \\ & & d_2 \downarrow & & \\ & & H^*(G/T) & & \end{array}$$

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the polynomials $e_i, \alpha_j, \beta_j \in \langle \omega_1, \cdots, \omega_n \rangle$ yield the integral cohomology classes

$$\varrho_k := \kappa \circ \widetilde{\varphi}(e_i) \in H^*(G), \ k = \deg e_i - 1$$

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the polynomials $e_i, \alpha_j, \beta_j \in \langle \omega_1, \cdots, \omega_n \rangle$ yield the integral cohomology classes

$$\varrho_k := \kappa \circ \widetilde{\varphi}(e_i) \in H^*(G), \ k = \deg e_i - 1$$

$$\varrho_k := \kappa \circ \widetilde{\varphi}(p_j\beta_j - y_j^{k_j-1}\alpha_j) \in H^*(G), \ k = \deg \beta_j - 1.$$

Type III. Finally, for a prime $p \in \{2,3,5\}$ and a multi-index $I \subset \{1, \cdots, m\}$ with $p_j = p$ for all $j \in I$ we set

$$\mathcal{C}_I^{(p)} := \delta_p(\cup_{t \in I} \zeta_t) \in H^*(G),$$

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- $\delta_p : H^r(G;\mathbb{F}_p) \to H^{r+1}(G)$ is the Bockstein homomorphism;
- letting κ_p and φ_p be the analogue of the maps κ and φ in the characteristic p, then

$$\zeta_t = \kappa_p \circ \varphi_p(\alpha_t) \in H^*(G;\mathbb{F}_p),$$

with $\deg \zeta_t = \deg \alpha_t - 1$.

Recall that integral cohomology of any finite dimensional cell complex X admits the decomposition

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In term of the three types x_s, ϱ_t and $\mathcal{C}_I^{(p)}$ of integral cohomology classes uniformly constructed for all Lie groups G, I will present, in accordance to above formula, the cohomology rings $H^*(G)$ of the five exceptional Lie groups

$$G = G_2, F_4, E_6, E_7, E_8.$$

Theorem 8

The ring $H^*(G_2)$ has the presentation

$$H^*(G_2) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}) \oplus \tau_2(G_2)$$

,

where

- $\tau_2(G_2) = \mathbb{F}_2[x_6]^+ / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3)$
- $\varrho_3^2 = x_6,\ x_6 \varrho_{11} = 0,$

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- $\varrho_3^2 = x_6,\ x_6\varrho_{11} = 0,$

on which the reduced co-product ψ is given by

- $\{\varrho_3, x_6\} \subset \mathcal{P}(G_2),$
- $\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5).$

Theorem 9

The ring $H^*(F_4)$ has the presentation

$$H^*(F_4) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}, \varrho_{15}, \varrho_{23}) \oplus \tau_2(F_4) \oplus \tau_3(F_4)$$

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- $\tau_3(F_4) = \mathbb{F}_3[x_8]^+ / \left\langle x_8^3 \right\rangle \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{11}, \varrho_{15})$
- $\varrho_3^2 = x_6,\ x_6 \varrho_{11} = 0,\ x_8 \varrho_{23} = 0,$

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on which the reduced co-product ψ is given by

- $\{\varrho_3, x_6, x_8\} \subset \mathcal{P}(F_4),$
- $\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3,$
- $\psi(\varrho_{15}) = -\delta_3(\zeta_7 \otimes \zeta_7),$
- $\psi(\varrho_{23}) = \delta_3(\zeta_7 \otimes \zeta_7 x_8 - \zeta_7 x_8 \otimes \zeta_7).$

Theorem 10

The ring $H^*(E_6)$ has the presentation

$$H^*(E_6) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_9, \varrho_{11}, \varrho_{15}, \varrho_{17}, \varrho_{23}) \oplus \tau_2(E_6) \oplus \tau_3(E_6)$$

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- $\tau_3(E_6) = \mathbb{F}_3[x_8]^+ / \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_9, \varrho_{11}, \varrho_{15}, \varrho_{17})$;
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- $\{\varrho_3, \varrho_9, \varrho_{17}, x_6, x_8\} \subset \mathcal{P}(E_6)$;
- $\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3$;
- $\psi(\varrho_{15}) = x_6 \otimes \varrho_9 - \delta_3(\zeta_7 \otimes \zeta_7)$;
- $\psi(\varrho_{23}) = x_6 \otimes \varrho_{17} + \delta_3(\zeta_7 x_8 \otimes \zeta_7 - \zeta_7 \otimes \zeta_7 x_8)$.

Theorem 11

The ring $H^*(E_7)$ has the presentation

$$\Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}, \varrho_{15}, \varrho_{19}, \varrho_{23}, \varrho_{27}, \varrho_{35}) \oplus \tau_2(E_7) \oplus \tau_3(E_7)$$

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where

- $\tau_2(E_7) = \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, \mathcal{C}_I]^+}{\langle x_6^2, x_{10}^2, x_{18}^2, \mathcal{D}_J, \mathcal{R}_K, \mathcal{S}_{I,J}, \mathcal{H}_{t,L} \rangle} \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{15}, \varrho_{23}, \varrho_{27})$
- $\tau_3(E_7) = \frac{\mathbb{F}_3[x_8]^+}{\langle x_8^3 \rangle} \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{11}, \varrho_{15}, \varrho_{19}, \varrho_{27}, \varrho_{35})$
- $\varrho_3^2 = x_6, x_8\varrho_{23} = 0,$

and where $t \in e(E_7, 2) = \{3, 5, 9\}, \quad I, J, L \subseteq e(E_7, 2), \quad |I|, |J| \geq 2.$

Theorem 12

The ring $H^*(E_8)$ has the presentation

$$H^*(E_8) = \Delta_{\mathbb{Z}}(\varrho_3, \varrho_{15}, \varrho_{23}) \otimes \Lambda_{\mathbb{Z}}(\varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47}, \varrho_{59}) \bigoplus_{p=2,3,5} \tau_p(E_8)$$

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where

- $\tau_2(E_8) = \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, x_{30}, \mathcal{C}_I]^+}{\langle x_6^8, x_{10}^4, x_{18}^2, x_{30}^2, \mathcal{D}_J, \mathcal{R}_K, \mathcal{S}_{I,J}, \mathcal{H}_{t,L} \rangle} \otimes \Delta_{\mathbb{F}_2}(\varrho_3, \varrho_{15}, \varrho_{23}) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{27})$
- $\tau_3(E_8) = \frac{\mathbb{F}_3[x_8, x_{20}, \mathcal{C}_{\{4,10\}}]^+}{\left\langle x_8^3, x_{20}^3, x_8^2 x_{20}^2 \mathcal{C}_{\{4,10\}}, \mathcal{C}_{\{4,10\}}^2 \right\rangle} \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{15}, \varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47})$
- $\tau_5(E_8) = \frac{\mathbb{F}_5[x_{12}]^+}{\langle x_{12}^5 \rangle} \otimes \Lambda_{\mathbb{F}_5}(\varrho_3, \varrho_{15}, \varrho_{23}, \varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47})$
- $\varrho_3^2 = x_6,\ \varrho_{15}^2 = x_{30},\ \varrho_{23}^2 = x_6^6 x_{10},\ x_{2s}\varrho_{3s-1} = 0$, *for* $s = 4, 5$
  $x_8\varrho_{59} = x_{20}^2 \mathcal{C}_{\{4,10\}},\ x_{20}\varrho_{23} = x_8^2 \mathcal{C}_{\{4,10\}},\ x_{12}\varrho_{59} = 0,$

and where $t \in e(E_8, 2) = \{3, 5, 9, 15\},\ K, I, J, L \subseteq e(E_8, 2),\ |I|, |J| \geq 2,\ |K| \geq 3.$

Proof of the Theorems (the main idea): Since the generators of the ring $H^*(G)$

$$x_i, \varrho_k, \mathcal{C}_I$$

are constructed explicitly from the polynomials in the Schubert classes,

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End of the proof.

Our next project is:
Construct the integral cohomology of the classifying space BG from Schubert presentation of the ring $H^*(G/T)$ using the fibration:

$$G/T \hookrightarrow BT \to BG$$

References. The examples are taken from the first paper in the list:

- Duan and Zhao, Schubert calculus and cohomology of Lie groups, Part I. 1-connected Lie groups, arXiv:0711.2541.
- Duan and Zhao, Schubert calculus and the Hopf algebra structures of exceptional Lie groups, Forum Mathematicum, Volume 26, 2014.
- Duan, Schubert calculus and cohomology of Lie groups, Part II, Compact Lie groups, arXiv:1502.00410.

Thanks!