Schubert calculus and cohomology of Lie groups

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SUSTECH

June 6, 2018
E. Cartan (1929): Determine the cohomology $H^*(G; \mathbb{F})$ (with $\mathbb{F} = \mathbb{R}$, $\mathbb{F}_p$ or $\mathbb{Z}$) of compact Lie groups $G$. 
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Backgrounds of the problem:

- Cartan has completed his classification on the Lie groups and symmetric spaces (1928)
- Lefschetz has secured the foundation of the homology and cohomology theory for the cell-complexes (1927).
The contents of the talk

1. Classical methods and the remaining problem
2. Schubert calculus
3. The integral cohomology of Lie groups (Duan, Zhao)
Classical methods and the remaining problem

Theorem 1

Every compact, connected and finite dimensional Lie group $G$ has the canonical form:

$$G = (G_1 \times \cdots \times G_k \times T^r)/K,$$

where

1. each $G_i$ is one of the following 1-connected simple Lie groups: $SU(n)$, $Sp(n)$, $Spin(n)$, $G_2$, $F_4$, $E_6$, $E_7$, $E_8$;
2. $T^r = S^1 \times \cdots \times S^1$ is the $r$-dimensional torus group;
3. $K$ is a finite subgroup of the center of $G_1 \times \cdots \times G_k \times T^r$.

Therefore, we shall adopt the convention in this talk: "the Lie groups $G$ under our consideration are the 1-connected and simple ones listed above."
Classical methods and the remaining problem

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Based on the cell decompositions on these groups Pontryagin (1938) obtained that

**Theorem 2**

The Poincare polynomials of the classical groups are

- $P_t(Spin(2n + 1)) = (1 - t^3)(1 - t^7) \cdots (1 - t^{4n-1});$
- $P_t(Spin(2n)) = (1 - t^3)(1 - t^7) \cdots (1 - t^{4n-5})(1 - t^{2n-1});$
- $P_t(SU(n)) = (1 - t^3)(1 - t^5) \cdots (1 - t^{2n-1});$
- $P_t(Sp(n)) = (1 - t^3)(1 - t^7) \cdots (1 - t^{4n-1}).$
A remarkable change began with Hopf in 1941, who studied the following problem:

**Problem**

*Classify those algebras* $\alpha : A \otimes A \to A$ *over the reals that can be furnished with a co-product* $\beta : A \to A \otimes A$. 

*Hopf algebra*
A remarkable change began with Hopf in 1941, who studied the following problem:

Problem

Classify those algebras \( \alpha : A \otimes A \to A \) over the reals that can be furnished with a co-product \( \beta : A \to A \otimes A \).

The group of multiplication \( \mu : G \times G \to G \) on \( G \) induces a co-product

\[
\beta = \mu^* : H^*(G; \mathbb{R}) \to H^*(G \times G; \mathbb{R}) = H^*(G; \mathbb{R}) \otimes H^*(G; \mathbb{R})
\]

that furnishes the cohomology \( H^*(G; \mathbb{R}) \) with the structure of a \textbf{Hopf algebra}. 
Classical methods and the remaining problem

In term of the co-product $\beta : A \rightarrow A \otimes A$ Hopf introduced the set of the primative elements in the algebra $A$

$$P(A) = \{ a \in A \mid \beta(a) = a \otimes 1 \oplus 1 \otimes a \}.$$
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Since this is a real vector space over $R$ one can take a basis

$$x_1, \ldots, x_n; y_1, \ldots, y_m$$

for $P(A)$ with $\text{deg}(x_i) = \text{even}$ and $\text{deg}(y_i) = \text{odd}$. 
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for $P(A)$ with $\text{deg}(x_i) = \text{even}$ and $\text{deg}(y_i) = \text{odd}$. With these notation Hopf proved that

**Theorem 3**

$$A = \mathbb{R}[x_1, \ldots, x_n] \otimes \wedge_{\mathbb{R}}(y_1, \ldots, y_m)$$
Classical methods and the remaining problem

**Corollary**

If $G$ is a finite dimensional Lie group, $H^*(G; \mathbb{R}) = \wedge_{\mathbb{R}}(y_1, \cdots, y_m)$.

**Corollary**

Conjecture of Cartan (1936): $H^*(G; \mathbb{R}) = H^*(S^{2n_1-1} \times \cdots \times S^{2n_k-1}; \mathbb{R})$.
Classical methods and the remaining problem

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Corollary

Yan (1949): Let $G$ be an exceptional Lie group. Then

1. $H^*(G_2; \mathbb{R}) = \bigwedge_{\mathbb{R}} (y_3, y_{11})$
2. $H^*(F_4; \mathbb{R}) = \bigwedge_{\mathbb{R}} (y_3, y_{11}, y_{15}, y_{23})$
3. $H^*(E_6; \mathbb{R}) = \bigwedge_{\mathbb{R}} (y_3, y_9, y_{11}, y_{15}, y_{17}, y_{23})$
4. $H^*(E_7; \mathbb{R}) = \bigwedge_{\mathbb{R}} (y_3, y_{11}, y_{15}, y_{19}, y_{23}, y_{27}, y_{35})$
5. $H^*(E_8; \mathbb{R}) = \bigwedge_{\mathbb{R}} (y_3, y_{15}, y_{23}, y_{27}, y_{35}, y_{39}, y_{47}, y_{59})$
Borel (1953) initiated the work to compute the cohomology $H^*(G; \mathbb{F}_p)$ of Lie groups $G$ with coefficients in a finite field $\mathbb{F}_p$. 
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He began with the following classifying result:

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*Let $A$ be a finitely generated Hopf over a finite field $\mathbb{F}_p$. Then*

$$A = B(x_1) \otimes \cdots \otimes B(x_n)$$
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where each $B(x_i)$ is one of the monogenic Hopf algebra over $\mathbb{F}_p$:

<table>
<thead>
<tr>
<th>$B(x_i)$</th>
<th>$\text{deg}(x_i)$ odd</th>
<th>$\text{deg}(x_i)$ even</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \neq 2$</td>
<td>$\Lambda_{\mathbb{F}_p}(x_i)$</td>
<td>$\mathbb{F}_p(x_i)/(x_i^{p^r})$</td>
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Classical methods and the remaining problem

Based on Borel’s classification on the Hopf algebra, the cohomologies \( H^*(G; \mathbb{F}_p) \) have been calculated case by case:

- Borel (1953) computed \( H^*(G_2; \mathbb{F}_2), H^*(F_4; \mathbb{F}_2) \);
- Araki (1960) computed \( H^*(F_4; \mathbb{F}_3) \);
- Toda, Kono, Mimura, Shimada (1973, 75, 76) obtained \( H^*(E_i; \mathbb{F}_2), i = 6, 7, 8 \);
- Kono, Mimura (1975, 1977) obtained \( H^*(E_i; \mathbb{F}_3), i = 6, 7, 8 \);
- Kono (1977) obtained \( H^*(E_8; \mathbb{F}_5) \)

Example

The cohomology algebra \( H^*(E_8; \mathbb{F}_p) \) of the exceptional Lie group \( E_8 \) is:

- If \( p = 2 \):
  \( \mathbb{F}_2 \langle \alpha_3, \alpha_5, \alpha_9, \alpha_{15} \rangle \otimes \Lambda \mathbb{F}_2(\alpha_{17}, \alpha_{23}, \alpha_{27}, \alpha_{29}) \)
- If \( p = 3 \):
  \( \mathbb{F}_3 \langle x_8, x_{20} \rangle / \langle x_3, x_{20} \rangle \otimes \Lambda \mathbb{F}_3(\zeta_3, \zeta_7, \zeta_{15}, \zeta_{19}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47}) \)
- If \( p = 5 \):
  \( \mathbb{F}_5 \langle x_{12} \rangle / \langle x_5, x_{12} \rangle \otimes \Lambda \mathbb{F}_5(\zeta_3, \zeta_{11}, \zeta_{15}, \zeta_{23}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47}) \)
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The cohomology algebra $H^*(E_8; \mathbb{F}_p)$ of the exceptional Lie group $E_8$ is
- If $p = 2$ : $\mathbb{F}_2[\alpha_3, \alpha_5, \alpha_9, \alpha_{15}] \langle \alpha_3^{16}, \alpha_5^8, \alpha_9^4, \alpha_{15}^4 \rangle \otimes \Lambda_{\mathbb{F}_2}(\alpha_{17}, \alpha_{23}, \alpha_{27}, \alpha_{29})$
- If $p = 3$ : $\mathbb{F}_3[x_8, x_{20}] / \langle x_8^3, x_{20}^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\zeta_3, \zeta_7, \zeta_{15}, \zeta_{19}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47})$;
- $p = 5$ : $\mathbb{F}_5[x_{12}] / \langle x_{12}^5 \rangle \otimes \Lambda_{\mathbb{F}_5}(\zeta_3, \zeta_{11}, \zeta_{15}, \zeta_{23}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47})$. 
Classical methods and the remaining problem

The problem we shall study is:

**Problem**

Find a **unified construction** of the integral cohomology ring $H^*(G)$ of compact Lie groups $G$, in particular, of $G = G_2, F_4, E_6, E_7, E_8$. 
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1. The integral cohomology $H^*(G)$ for $G = SU(n), Sp(n), Spin(n)$ has been determined by Borel (1952) and Pittie (1991).

2. The integral cohomology $H^*(G)$ fails to be an Hopf ring:

$$
\beta = \mu^* : H^*(G) \to H^*(G \times G) \cong H^*(G) \otimes H^*(G) \oplus Ext(H^*(G), H^*(G))
$$

1) the isomorphism $\cong$ given by the Kunneth formula is additive, but is not multiplicative;
2) the appearance of the summand $Ext(H^*(G), H^*(G))$
Schubert calculus

Schubert calculus is the intersection theory founded by Poncellet, Charles, Schubert, ⋯ , in the 19th century, together with its applications to enumerative geometry:

Example

Given 8 quadrics in the space $\mathbb{P}^3$, find the number of conics tangent to all 8. (4,407,296)

Given 9 quadrics in the space $\mathbb{P}^3$, find the number of quadrics tangent to all 9. (666,841,088)

Given 12 quadrics in the space $\mathbb{P}^3$, find the number of twisted cubic space curves tangent to all 12. (5,819,539,783,680).
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References:

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References

**Schubert calculus**

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van der Waerden, 1929: “the common task of all the enumerative methods is to compute the intersection numbers of subvarieties in the cohomology theory founded by Lefschetz in 1927”

A. Weil, 1948: ”The classical Schubert calculus amounts to the determination of cohomology rings $H^*(G/P)$ of flag manifolds $G/P$” where $G$ is a Lie group and $P \subset G$ is a parabolic subgroup.

References

- van der Waerden, Topologische Begrundung des Kalkuls der abzählenden Geometrie, 1929.
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Chevalley announced in 1958 that the Schubert varieties \( \{X_w \subset G/P \mid w \in W\} \) on a flag manifold \( G/P \) provide a cell decomposition

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G/P = \bigcup_{w \in W} X_w, \quad \dim_R X_w = 2l(w), \quad W = W(G)/W(P).
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and the **fundamental problem of Schubert calculus:**

**Problem**

Determine the structure constants \( c_{u,v}^w \in \mathbb{Z} \) required to expand the product of two arbitrary Schubert classes

\[
s_u \cup s_v = \sum c_{u,v}^w \cdot s_w.
\]
Schubert calculus

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Example

*The table of structure constants of the variety of complete conics on $P^3$ (from Schubert’s book in 1879)*

![Table of structure constants](Image)
Schubert calculus

Example

The table of structure constants of the variety of complete quadrics on $P^3$
(from Schubert’s book in 1879)

"The fundamental problem which occupies Schubert is to express the product of two of these symbols in terms of others linearly. He succeeds in part."

The solution to the "fundamental problem of Schubert calculus" has been obtained by Duan in the works


**Theorem 6**

If \( u, v \in W \) with \( l(w) = l(u) + l(v) \) then

\[
c^w_{u,v} = T_{A_w} \left[ \left( \sum_{|L| = l(u), \sigma_L = u} x_L \right) \left( \sum_{|K| = l(v), \sigma_K = v} x_K \right) \right]
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This formula

- express the structure constant \( c^w_{u,v} \) as a polynomial in the Cartan numbers of the Lie group \( G \);
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This formula

- express the structure constant \( c_w^{u,v} \) as a polynomial in the Cartan numbers of the Lie group \( G \);
- applies uniformly to all flag manifold \( G/P \).
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Based on the formula, a computer package entitled ”The Chow ring of flag varieties” has been composed in the following works

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whose function can be described as follows:

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**Input:** The Cartan matrix $A = (a_{ij})_{n \times n}$ of the Lie group $G$, and a subset $I \subseteq \{1, 2, \cdots, n\}$ (to specify the parabolic subgroup $P \subset G$)
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**Output:**

- A minimal set of Schubert classes on $G/P$ that generates the cohomology $H^*(G/P)$ multiplicatively;
- A presentation of the ring $H^*(G/P)$ in these generators.
Schubert calculus

Example

\begin{align*}
H^*({G_2}/T) &= \mathbb{Z}[\omega_1, \omega_2, y_3]/\langle \rho_2, r_3, r_6 \rangle, \text{ where} \\
\rho_2 &= 3\omega_1^2 - 3\omega_1\omega_2 + \omega_2^2; \\
r_3 &= 2y_3 - \omega_1^3; \\
r_6 &= y_3^2.
\end{align*}

(5.1)

\begin{align*}
H^*({F_4}/T) &= \mathbb{Z}[\omega_1, \omega_2, \omega_3, \omega_4, y_3, y_4]/\langle \rho_2, \rho_4, r_3, r_6, r_8, r_{12} \rangle \text{ where} \\
\rho_2 &= c_2 - 4\omega_1^2; \\
\rho_4 &= 3y_4 + 2\omega_1y_3 - c_4; \\
r_3 &= 2y_3 - \omega_1^3; \\
r_6 &= y_3^2 + 2c_6 - 3\omega_1^2y_4; \\
r_8 &= 3y_4^2 - \omega_1^2c_6; \\
r_{12} &= y_4^3 - c_6^2.
\end{align*}

(5.2)

\begin{align*}
H^*({E_6}/T) &= \mathbb{Z}[\omega_1, \ldots, \omega_6, y_3, y_4]/\langle \rho_2, \rho_3, \rho_4, \rho_5, r_6, r_8, r_9, r_{12} \rangle, \text{ where} \\
\rho_2 &= 4\omega_2^2 - c_2; \\
\rho_3 &= 2y_3 + 2\omega_2^3 - c_3; \\
\rho_4 &= 3y_4 + \omega_2^4 - c_4; \\
\rho_5 &= 2\omega_2^2y_3 - \omega_2c_4 + c_5; \\
r_6 &= y_3^2 - \omega_2c_5 + 2c_6; \\
r_8 &= 3y_4^2 - 2c_5y_3 - \omega_2^2c_6 + \omega_2^3c_5; \\
r_9 &= 2y_3c_6 - \omega_2^2c_6; \\
r_{12} &= y_4^3 - c_6^2.
\end{align*}

(5.3)
For a Lie group $G$ with a maximal torus $T$, let $\{\omega_1, \cdots, \omega_n \in H^2(G/T)\}$ be the set of fundamental dominant weights of $G$.

**Theorem 7**

*The ring $H^*(G/T)$ has the presentation*

$$H^*(G/T) = \mathbb{Z}[\omega_1, \cdots, \omega_n, y_1, \cdots, y_m] / \langle e_i, f_j, g_j \rangle_{1 \leq i \leq n-m; 1 \leq j \leq m}$$

*where the polynomial relations $e_i, f_j, g_j$ have the following properties:*

1. For each $1 \leq i \leq n - m$, $e_i \in \langle \omega_1, \cdots, \omega_n \rangle$;
2. For each $1 \leq j \leq m$

   $$f_j = p_j \cdot y_j + \alpha_j, \quad g_j = y_j^{k_j} + \beta_j,$$

*with $p_j \in \{2, 3, 5\}$ and $\alpha_j, \beta_j \in \langle \omega_1, \cdots, \omega_n \rangle$.\[\]
Borel has shown in 1953 that, over the field $\mathbb{R}$ of reals

$$H^*(G/T; \mathbb{R}) = \mathbb{R}[\omega_1, \cdots, \omega_n]/\mathbb{R}[\omega_1, \cdots, \omega_n]^{W^+}.$$ 

But it fails if we replace $\mathbb{R}$ by the ring $\mathbb{Z}$ or by a finite field;
Schubert calculus

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But it fails if we replace $\mathbb{R}$ by the ring $\mathbb{Z}$ or by a finite field;

- We shall see in the coming section that,
  1. the set \( \{y_1, \cdots, y_m\} \) of Schubert classes on \( G/T \), and
  2. the polynomials \( e_i, \alpha_j, \beta_j \) in the Schubert classes \( \omega_1, \cdots, \omega_n, y_1, \cdots, y_m \) on \( G/T \)

are just what requested to construct the integral cohomology rings \( H^*(G) \) uniformly for all Lie group \( G \).
Cohomology of Lie groups

For a Lie group $G$ with a maximal torus $T$ consider the fibration

$$T \hookrightarrow G \xrightarrow{\pi} G/T.$$
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Our construction uses the Leray Serre spectral sequence $\{E^r_{\ast,\ast}(G), d_r\}$ in which one has

- $E^{\ast,\ast}_2(G) = H^\ast(G/T) \otimes H^\ast(T)$;
- $d_2(x \otimes t) = x \cup \tau(t) \otimes 1$, $x \in H^\ast(G/T)$, $t \in H^1(T)$. 

Added to this we have three crucial facts:

- The factor ring $H^\ast(G/T)$ of $E^{\ast,\ast}_2(G)$ has been made explicit in Duan+Zhao, LMS.J.Comput.Math. 2015;
- The Borel transgression $\tau: H^1(T) \rightarrow H^2(G/T)$ has been decided by Duan, in Homology, Homotopy Appl. 2018;
- $E^{\ast,\ast}_3(G) = H^\ast(G)$ (conjectured by Marlin in 1989).
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For a Lie group $G$ with a maximal torus $T$ consider the fibration

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Combining Schubert presentation of the ring $H^*(G/T)$ with the term \( \{E_2^*, *(G), d_2\} \) we construct below three types of explicit elements

\[ x_i, \varphi_k, C_l \in H^*(G) \]

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Combining Schubert presentation of the ring $H^*(G/T)$ with the term \{\(E_2^\ast, \ast(G), d_2\)\} we construct below three types of explicit elements

\[x_i, \varrho_k, C_l \in H^*(G)\]

that generate the ring $H^*(G)$ multiplicatively:

**Type I.** In view of the quotient map $\pi : G \to G/T$, the set \(\{y_1, \cdots, y_m\}\) of special Schubert classes on $G/T$ yields directly the integral cohomology classes of the group $G$

\[x_{\deg y_i} := \pi^*(y_i) \in H^*(G), 1 \leq i \leq m\]
Type II. In view of the ready made maps

\[ \langle \omega_1, \cdots, \omega_n \rangle \xrightarrow{\varphi} E^{2k,1}_2(G), \quad E^{2k,1}_3(G) \xrightarrow{\kappa} H^{2k+1}(G) \]

\[ \varphi(g_1 \omega_1 + \cdots + g_n \omega_n) = g_1 \otimes t_1 + \cdots + g_n \otimes t_n \]

and the commutative diagrams

\[
\begin{array}{ccc}
E^{*,1}_3(G) & \xrightarrow{\kappa} & H^*(G) \\
\varphi \uparrow & & \downarrow \\
\langle \omega_1, \cdots, \omega_n \rangle & \xrightarrow{\varphi} & E^{*,1}_2(G) \\
& & d_2 \downarrow \\
& & H^*(G/T)
\end{array}
\]
Type II. In view of the ready made maps

\[ \langle \omega_1, \cdots, \omega_n \rangle \xrightarrow{\varphi} E_{2k,1}^2(G), \quad E_{2k,1}^3(G) \xrightarrow{\kappa} H^{2k+1}(G) \]

\[ \varphi(g_1\omega_1 + \cdots + g_n\omega_n) = g_1 \otimes t_1 + \cdots + g_n \otimes t_n \]

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\[ E_{3,1}^*, (G) \xrightarrow{\kappa} H^*(G) \]

\[ \varphi \uparrow \downarrow \tilde{\varphi} \]

\[ \langle \omega_1, \cdots, \omega_n \rangle \xrightarrow{\varphi} E_{2,1}^*(G) \]

\[ d_2 \downarrow \]

\[ H^*(G/T) \]

the polynomials \( e_i, \alpha_j, \beta_j \in \langle \omega_1, \cdots, \omega_n \rangle \) yield the integral cohomology classes

\[ \varrho_k := \kappa \circ \tilde{\varphi}(e_i) \in H^*(G), \quad k = \deg e_i - 1 \]
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and the commutative diagrams

$$\begin{array}{ccc}
E^*_3(G) & \xrightarrow{\kappa} & H^*(G) \\
\sim \varphi & \uparrow & \\
\langle \omega_1, \cdots, \omega_n \rangle & \xrightarrow{\varphi} & E^*_2(G) \\
 & d_2 \downarrow & \\
 & H^*(G/T) & 
\end{array}$$

the polynomials $$e_i, \alpha_j, \beta_j \in \langle \omega_1, \cdots, \omega_n \rangle$$ yield the integral cohomology classes

$$\varrho_k := \kappa \circ \tilde{\varphi}(e_i) \in H^*(G), \quad k = \deg e_i - 1$$

$$\varrho_k := \kappa \circ \tilde{\varphi}(p_j \beta_j - y_j^{k_j - 1} \alpha_j) \in H^*(G), \quad k = \deg \beta_j - 1.$$
Type III. Finally, for a prime $p \in \{2, 3, 5\}$ and a multi-index $I \subset \{1, \cdots, m\}$ with $p_j = p$ for all $j \in I$ we set

$$C^{(p)}_I := \delta_p(\bigcup_{t \in I} \zeta_t) \in H^*(G),$$

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Type III. Finally, for a prime $p \in \{2, 3, 5\}$ and a multi-index $I \subset \{1, \cdots, m\}$ with $p_j = p$ for all $j \in I$ we set

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where

- $\delta_p : H^r(G; F_p) \rightarrow H^{r+1}(G)$ is the Bockstein homomorphism;
**Type III.** Finally, for a prime $p \in \{2, 3, 5\}$ and a multi-index $I \subset \{1, \cdots, m\}$ with $p_j = p$ for all $j \in I$ we set

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where

- $\delta_p : H^r(G; \mathbb{F}_p) \to H^{r+1}(G)$ is the Bockstein homomorphism;
- letting $\kappa_p$ and $\varphi_p$ be the analogue of the maps $\kappa$ and $\varphi$ in the characteristic $p$, then

$$\zeta_t = \kappa_p \circ \varphi_p(\alpha_t) \in H^*(G; \mathbb{F}_p),$$

with $\deg \zeta_t = \deg \alpha_t - 1$. 
Recall that integral cohomology of any finite dimensional cell complex $X$ admits the decomposition

$$H^*(X) = F(X) \oplus_p \tau_p(X)$$

where the sum $\oplus$ is over all prime $p$, and where
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- the ideal $\tau_p(X)$ is a module over $F(X): F(X) \times \tau_p(X) \to \tau_p(X)$.

In term of the three types $x_s$, $\varrho_t$ and $C^{(p)}_i$ of integral cohomology classes uniformly constructed for all Lie groups $G$, I will present, in accordance to above formula, the cohomology rings $H^*(G)$ of the five exceptional Lie groups

$$G = G_2, F_4, E_6, E_7, E_8.$$
Theorem 8

The ring $H^*(G_2)$ has the presentation

$$H^*(G_2) = \Delta_\mathbb{Z}(\varrho_3) \otimes \Lambda_\mathbb{Z}(\varrho_{11}) \oplus \tau_2(G_2),$$

where

- $\tau_2(G_2) = F_2[x_6]^{+}/\langle x_6^2 \rangle \otimes \Delta_{F_2}(\varrho_3)$
- $\varrho_3^2 = x_6$, $x_6 \varrho_{11} = 0$. 
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- $\varrho_3^2 = x_6$, $x_6 \varrho_{11} = 0$,

on which the reduced co-product $\psi$ is given by

- $\{ \varrho_3, x_6 \} \subset \mathcal{P}(G_2)$,
- $\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5)$. 
Theorem 9

The ring $H^*(F_4)$ has the presentation

$$H^*(F_4) = \Delta_\mathbb{Z}(\rho_3) \otimes \Lambda_\mathbb{Z}(\rho_{11}, \rho_{15}, \rho_{23}) \oplus \tau_2(F_4) \oplus \tau_3(F_4)$$
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where

- $\tau_2(F_4) = \mathbb{F}_2[x_6]^+ / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{11}, \varrho_{23})$
- $\tau_3(F_4) = \mathbb{F}_3[x_8]^+ / \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{11}, \varrho_{15})$
- $\varrho_3^2 = x_6$, $x_6 \varrho_{11} = 0$, $x_8 \varrho_{23} = 0$, $\varrho_3 = x_6$, $\varrho_{11} = 0$, $\varrho_{23} = 0$,
Theorem 9

The ring $H^*(F_4)$ has the presentation

$$H^*(F_4) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}, \varrho_{15}, \varrho_{23}) \oplus \tau_2(F_4) \oplus \tau_3(F_4)$$

where

- $\tau_2(F_4) = \mathbb{F}_2[x_6]^+ / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{15}, \varrho_{23})$
- $\tau_3(F_4) = \mathbb{F}_3[x_8]^+ / \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{11}, \varrho_{15})$
- $x_6^2 = x_6, \ x_6 \varrho_{11} = 0, \ x_8 \varrho_{23} = 0$,

on which the reduced co-product $\psi$ is given by

- $\{\varrho_3, x_6, x_8\} \subset P(F_4)$,
- $\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3$,
- $\psi(\varrho_{15}) = -\delta_3(\zeta_7 \otimes \zeta_7)$,
- $\psi(\varrho_{23}) = \delta_3(\zeta_7 \otimes \zeta_7 x_8 - \zeta_7 x_8 \otimes \zeta_7)$.
Theorem 10

The ring $H^*(E_6)$ has the presentation

$$H^*(E_6) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_9, \varrho_{11}, \varrho_{15}, \varrho_{17}, \varrho_{23}) \oplus \tau_2(E_6) \oplus \tau_3(E_6)$$

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where

- $\tau_2(E_6) = F_2[x_6]^+ / \langle x_6^2 \rangle \otimes \Delta_{F_2}(\rho_3) \otimes \Lambda_{F_2}(\rho_9, \rho_{15}, \rho_{17}, \rho_{23})$;
- $\tau_3(E_6) = F_3[x_8]^+ / \langle x_8^3 \rangle \otimes \Lambda_{F_3}(\rho_3, \rho_9, \rho_{11}, \rho_{15}, \rho_{17})$;
- $\rho_3^2 = x_6$, $x_6 \rho_{11} = 0$, $x_8 \rho_{23} = 0$.
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where

- $\tau_2(E_6) = \mathbb{F}_2[x_6] / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_9, \varrho_{15}, \varrho_{17}, \varrho_{23})$;
- $\tau_3(E_6) = \mathbb{F}_3[x_8] / \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_9, \varrho_{11}, \varrho_{15}, \varrho_{17})$;
- $\varrho_3^2 = x_6, \ x_6 \varrho_{11} = 0, \ x_8 \varrho_{23} = 0$

on which the reduced co-product $\psi$ is given by

- $\{\varrho_3, \varrho_9, \varrho_{17}, x_6, x_8\} \subset \mathcal{P}(E_6)$;
- $\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3$;
- $\psi(\varrho_{15}) = x_6 \otimes \varrho_9 - \delta_3(\zeta_7 \otimes \zeta_7)$;
- $\psi(\varrho_{23}) = x_6 \otimes \varrho_{17} + \delta_3(\zeta_7 x_8 \otimes \zeta_7 - \zeta_7 \otimes \zeta_7 x_8)$.  

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Theorem 11

The ring $H^*(E_7)$ has the presentation

$$\Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}, \varrho_{15}, \varrho_{19}, \varrho_{23}, \varrho_{27}, \varrho_{35}) \oplus \tau_2(E_7) \oplus \tau_3(E_7)$$
Theorem 11

The ring $H^*(E_7)$ has the presentation

$$\Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}, \varrho_{15}, \varrho_{19}, \varrho_{23}, \varrho_{27}, \varrho_{35}) \oplus \tau_2(E_7) \oplus \tau_3(E_7)$$

where

- $\tau_2(E_7) = \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, C_I]^+}{\langle x_6^2, x_{10}^2, x_{18}^2, D_J, R_K, S_I, J, H_t, L \rangle} \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{15}, \varrho_{23}, \varrho_{27})$
- $\tau_3(E_7) = \frac{\mathbb{F}_3[x_8]^+}{\langle x_8^3 \rangle} \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{11}, \varrho_{15}, \varrho_{19}, \varrho_{27}, \varrho_{35})$
- $\varrho_2^3 = x_6, x_8 \varrho_{23} = 0,$

and where $t \in e(E_7, 2) = \{3, 5, 9\}, \ I, J, L \subseteq e(E_7, 2), \ |I|, |J| \geq 2.$
Theorem 12

The ring $H^*(E_8)$ has the presentation

$$H^*(E_8) = \Delta \mathbb{Z}(\varrho_3, \varrho_{15}, \varrho_{23}) \otimes \Lambda \mathbb{Z}(\varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47}, \varrho_{59}) \bigoplus_{p=2,3,5} \tau_p(E_8)$$
Theorem 12

The ring $H^*(E_8)$ has the presentation

$$H^*(E_8) = \Delta_{\mathbb{Z}}(\varrho_3, \varrho_{15}, \varrho_{23}) \otimes \Lambda_{\mathbb{Z}}(\varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47}, \varrho_{59}) \oplus \tau_p(E_8)_{p=2,3,5}$$

where

- $\tau_2(E_8) = \frac{F_2[x_6, x_{10}, x_{18}, x_{30}, c_I]^+}{\langle x_6^8, x_{10}^4, x_{18}^2, x_{30}^2, D_J, R_K, S_I, J, H_t, L \rangle} \otimes \Delta_{F_2}(\varrho_3, \varrho_{15}, \varrho_{23}) \otimes \Lambda_{F_2}(\varrho_{27})$
- $\tau_3(E_8) = \frac{F_3[x_8, x_{20}, c_{\{4,10\}}]^+}{\langle x_8^3, x_{20}^3, x_8^2 x_{20}^2 c_{\{4,10\}}, c_{\{4,10\}}^2 \rangle} \otimes \Lambda_{F_3}(\varrho_3, \varrho_{15}, \varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47})$
- $\tau_5(E_8) = \frac{F_5[x_{12}]^+}{\langle x_{12}^5 \rangle} \otimes \Lambda_{F_5}(\varrho_3, \varrho_{15}, \varrho_{23}, \varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47})$
- $\varrho_3^2 = x_6, \varrho_{15}^2 = x_{30}, \varrho_{23}^2 = x_6^6 x_{10}, x_{2s} \varrho_{3s-1} = 0, \text{ for } s = 4, 5$
- $x_8 \varrho_{59} = x_8^2 c_{\{4,10\}}, x_{20} \varrho_{23} = x_8^2 c_{\{4,10\}}, x_{12} \varrho_{59} = 0$

and where $t \in e(E_8, 2) = \{3, 5, 9, 15\}, K, I, J, L \subseteq e(E_8, 2), |I|, |J| \geq 2, |K| \geq 3$. 
Proof of the Theorems (the main idea): Since the generators of the ring $H^*(G)$

$$x_i, \varrho_k, \mathcal{C}_l$$

are constructed explicitly from the polynomials in the Schubert classes,

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Proof of the Theorems (the main idea): Since the generators of the ring $H^*(G)$

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End of the proof.
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End of the proof.

Our next project is:
Construct the integral cohomology of the classifying space $BG$ from Schubert presentation of the ring $H^*(G/T)$ using the fibration:

$$G/T \hookrightarrow BT \rightarrow BG$$


References. The examples are taken from the first paper in the list:

- Duan and Zhao, Schubert calculus and cohomology of Lie groups, Part I. 1-connected Lie groups, arXiv:0711.2541.
- Duan and Zhao, Schubert calculus and the Hopf algebra structures of exceptional Lie groups, Forum Mathematicum, Volume 26, 2014.
Thanks!