Model categories and stable homotopy theory

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Iintroduction (continued)


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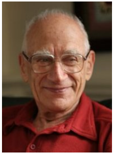
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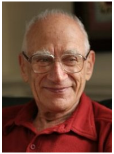

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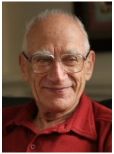

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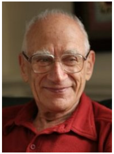

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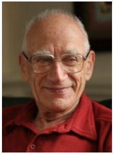

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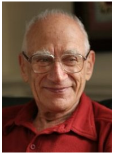

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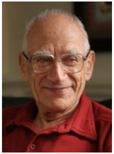

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We will see that the passage from strict equivalence to stable equivalence is a form of Bousfield localization.

Quillen model categories

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$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y, \end{array}$$

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$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & X \\ {\scriptstyle \text{cofibration } i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p \text{ trivial fibration}} \\ B & \xrightarrow[\quad g \quad]{} & Y, \end{array}$$

a morphism h (called a lifting) exists for i and p as indicated.

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cofibration / *trivial cofibration* (i); *trivial fibration* / *fibration* (p)

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cofibration / trivial fibration

trivial cofibration / fibration

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MC5 **Factorization axiom.** *Any morphism $f : X \to Y$ can be functorially factored in two ways as*

$$X \xrightarrow{f} Y$$

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Quillen model categories (continued)

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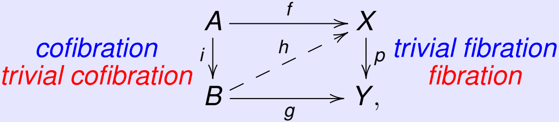

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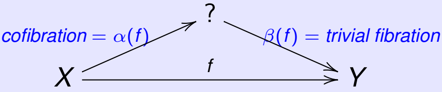

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Quillen model categories (continued)

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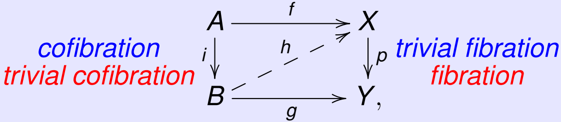

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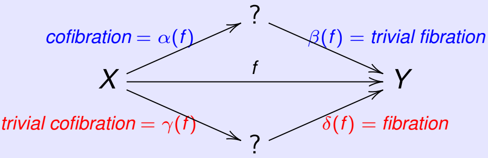

Model categories and stable homotopy theory

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Quillen model categories (continued)

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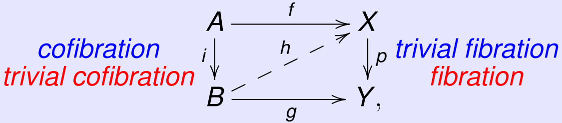

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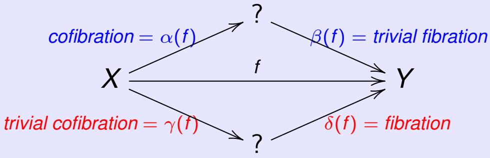

This last axiom is the hardest one to verify in practice.

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Two classical examples

Let $\mathcal{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces.

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Let $\mathcal{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups. Fibrations are Serre fibrations, that is maps $p : X \to Y$ with the right lifting property

$$\begin{array}{ccc} I^n & \xrightarrow{f} & X \\ {\scriptstyle j_n}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ I^{n+1} & \xrightarrow[g]{} & Y, \end{array} \qquad \text{for each } n \geq 0.$$

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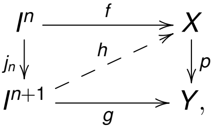

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Similar definitions can be made for $\mathcal{T}$, the category of pointed topological spaces and basepoint preserving maps.

Some definitions

Recall that we denote the initial and terminal objects of $\mathcal{M}$ by $\varnothing$ and $*$. When they are the same, we say that $\mathcal{M}$ is pointed.

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By **MC5**, for any object X, the unique maps $\varnothing \to X$ and $X \to *$ have factorizations

$$\varnothing \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

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Cofibrant generation

Example

In $\mathcal{T}op$, let

$$\mathcal{I} = \{ i_n : S^{n-1} \to D^n, n \geq 0 \} \text{ and } \mathcal{J} = \{ j_n : I^n \to I^{n+1}, n \geq 0 \} .$$

Cofibrant generation

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It is known that every (trivial) cofibration in $\mathcal{T}op$ can be derived from the ones in ($\mathcal{J}$) $\mathcal{I}$ by iterating certain elementary constructions.

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where X_+ denotes the space X with a disjoint base point.

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Cofibrant generation (continued)

Definition

A cofibrantly generated model category $\mathcal{M}$ is one with morphism sets $\mathcal{I}$ and $\mathcal{J}$ having similar properties to the ones in $\mathcal{T}op$. $\mathcal{I}$ ($\mathcal{J}$) is a generating set of (trivial) cofibrations.

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Bousfield localization

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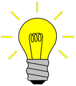

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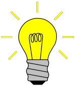
Bousfield localization



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Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects. This could make the fibrant replacement functor much more interesting.

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The hardest part of this is showing that the new classes of weak equivalences and fibrations, along with the original class of cofibrations, satisfy the second Factorization Axiom MC5. The proof involves some delicate set theory.

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Three examples of Bousfield localization

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Three examples of Bousfield localization

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3. Choose a generalized homology theory h_*. Define a map f to be a weak equivalence if $h_* f$ is an isomorphism. The resulting fibrant replacement functor is Bousfield localization with respect to h_*. One can do the same with the category of spectra, once we have the right model structure on it.

Enriched category theory

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Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

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Enriched category theory (continued)

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Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category

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Enriched category theory (continued)

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Definition

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$$\mathcal{C}_0(X, Y) = \mathcal{V}_0(\mathbf{1}, \mathcal{C}(X, Y)).$$

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Enriched category theory (continued)

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Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

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Enriched category theory (continued)

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In this language, an ordinary category is enriched over $\mathcal{S}et$.

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The symmetric monoidal categories $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{T}$ and $\mathcal{S}et_\Delta$ are each closed.

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Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$

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Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$.

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Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$. We will redefine it to be an enriched $\mathcal{T}$-valued functor on a small $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$.

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Definition

The indexing category $\mathscr{J}^{\mathbf{N}}$ has natural numbers $n \geq 0$ as objects with

$$\mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

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Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$. We will redefine it to be an enriched $\mathcal{T}$-valued functor on a small $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. This will make the structure maps built in to the functor. Maps between spectra will be enriched natural transformations.

Definition

The indexing category $\mathscr{J}^{\mathbf{N}}$ has natural numbers $n \geq 0$ as objects with

$$\mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

For $m \leq n \leq p$, the composition morphism

$$j_{m,n,p} : S^{p-n} \wedge S^{n-m} \to S^{p-m}$$

is the standard homeomorphism.

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$.

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Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n.

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$$\epsilon^X_{m,n} : \mathscr{J}^{\mathbf{N}}(m,n) \wedge X_m \to X_n.$$

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for $m \leq n$ we get the expected map $\Sigma^{n-m} X_m \to X_n$.

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This is the reason that the category of spectra $\mathcal{S}p$ defined in this way does not have a convenient smash product. This was a headache in the subject for decades!

Defining the smash product of spectra

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To repeat, spectra as originally defined do not have a convenient smash product.

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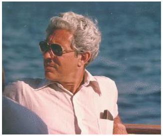

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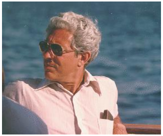

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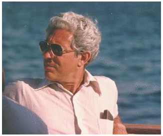

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Defining the smash product of spectra (continued)

The Hovey-Shipley-Smith approach was to enlarge the indexing category $\mathscr{J}^{\mathbf{N}}$ to make it into a symmetric monoidal category $\mathscr{J}^{\mathbf{\Sigma}}$.

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Defining the smash product of spectra (continued)

A symmetric spectrum is an $\mathcal{T}$-enriched functor $\mathscr{J}^{\boldsymbol{\Sigma}} \to \mathcal{T}$. Note that

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Defining the smash product of spectra (continued)

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Let $\mathcal{V}$ be a closed symmetric monoidal category, and let $\mathscr{J}$ be a symmetric monoidal $\mathcal{V}$-category.

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Let $\mathcal{V}$ be a closed symmetric monoidal category, and let $\mathcal{J}$ be a symmetric monoidal $\mathcal{V}$-category. Then the category of functors from $\mathcal{J}$ to $\mathcal{V}$ is also closed symmetric monoidal.

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$$\mathscr{J}^{\boldsymbol{\Sigma}} \times \mathscr{J}^{\boldsymbol{\Sigma}} \xrightarrow{X \times Y} \mathcal{T} \times \mathcal{T}$$

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$$\begin{array}{ccccc} & & (A,B) & \longmapsto & A \wedge B \\ \mathscr{J}^{\boldsymbol{\Sigma}} \times \mathscr{J}^{\boldsymbol{\Sigma}} & \xrightarrow{X \times Y} & \mathcal{T} \times \mathcal{T} & \xrightarrow{\wedge} & \mathcal{T} \end{array}$$

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$$(A, B) \longmapsto A \wedge B$$

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Defining the smash product of spectra (continued)

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Let $\mathcal{V}$ be a closed symmetric monoidal category, and let $\mathscr{J}$ be a symmetric monoidal $\mathcal{V}$-category. Then the category of functors from $\mathscr{J}$ to $\mathcal{V}$ is also also symmetric monoidal.

How does this work? Let X and Y be symmetric spectra. Then we have

$$\begin{array}{ccccc} & & (A,B) & \longmapsto & A \wedge B \\ \mathscr{J}^{\Sigma} \times \mathscr{J}^{\Sigma} & \xrightarrow{X \times Y} & \mathcal{T} \times \mathcal{T} & \xrightarrow{\wedge} & \mathcal{T} \\ (m,n) & \searrow^{+} & & & \\ & \longmapsto & \mathscr{J}^{\Sigma} & & \\ & & m+n & & \end{array}$$

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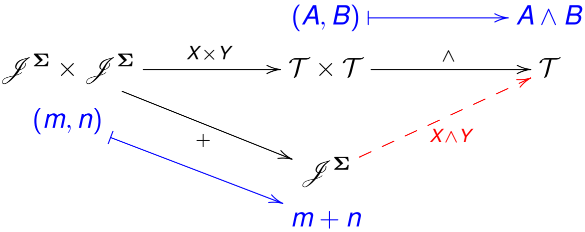

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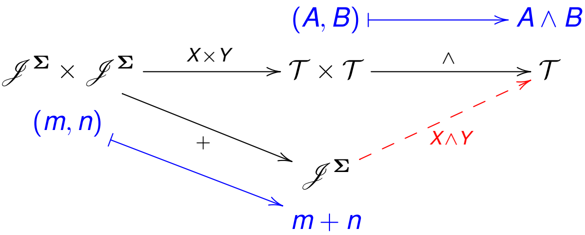

The red arrow is a left Kan extension,

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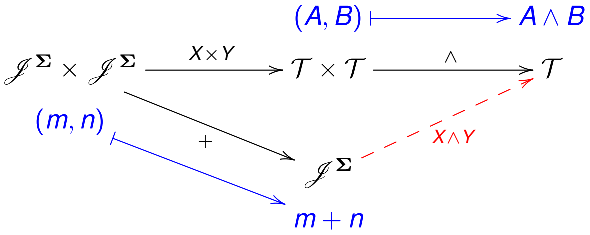

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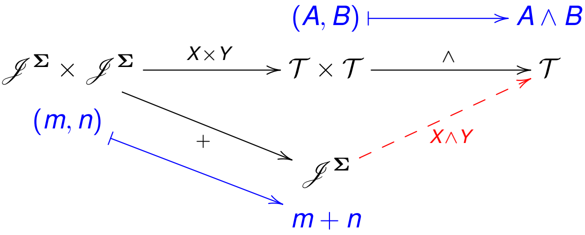

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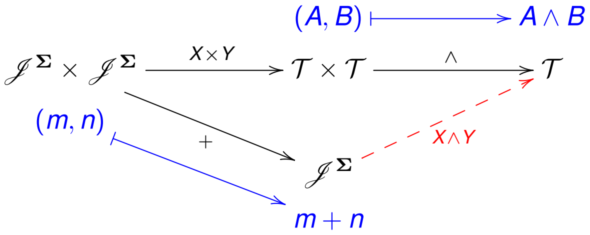

The red arrow is a left Kan extension, a categorical construction known to exist when the source category $\mathscr{J}^{\Sigma}$ is small and the target category $\mathcal{T}$ is cocomplete.

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Generalizations

The construction of symmetric spectra as functors

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Generalizations

The construction of symmetric spectra as functors from the symmetric monoidal $\mathcal{T}$-category $\mathscr{J}^{\Sigma}$

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Generalizations

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Generalizing the target category

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$$\mathcal{I}^G = \bigcup_{H \subseteq G} (G/H)_+ \wedge \mathcal{I}_+ \quad \text{and} \quad \mathcal{J}^G = \bigcup_{H \subseteq G} (G/H)_+ \wedge \mathcal{J}_+.$$

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We can replace $\mathscr{J}^{\Sigma}$ by an orthogonal analog $\mathscr{J}^{O}$, the Mandell-May category.

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In each case one has a smash product of spectra defined using the Day Convolution as before.

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Generalizing the model structure

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We have now seen four categories of spectra,

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We have now seen four categories of spectra, each defined as the category of enriched functors from an enriched indexing category $\mathscr{J}$ to a pointed topological model category $\mathcal{M}$. We denote such a functor category by $[\mathscr{J}, \mathcal{M}]$.

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In the structured cases one can show that for objects V' and V'' in $\mathscr{J}$,

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Definition

Let $\mathcal{I}$ and $\mathcal{J}$ be cofibrant generating sets for $\mathcal{M}$. In the projective model structure on the category of spectra $[\mathscr{J}, \mathcal{M}]$, the cofibrant generating sets are

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The Wth component of the map s_V is

$$j_{0,V,W} : \mathscr{J}(V, W) \wedge \mathscr{J}(0, V) \to \mathscr{J}(0, W),$$

a composition morphism in $\mathscr{J}$.

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