The Slice Spectral Sequence of a Height 4 Theory

XiaoLin Danny Shi

(Joint with Mike Hill, Guozhen Wang, and Zhouli Xu)

Harvard

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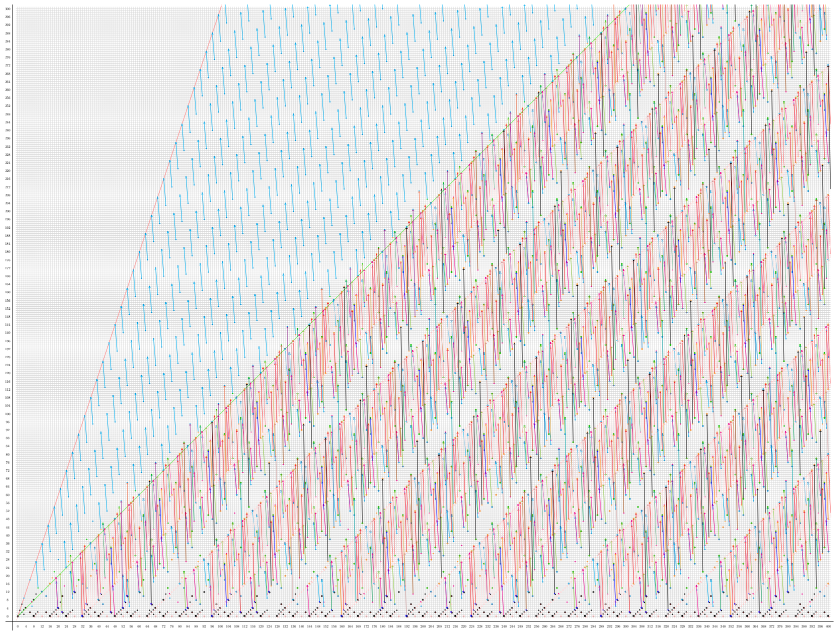

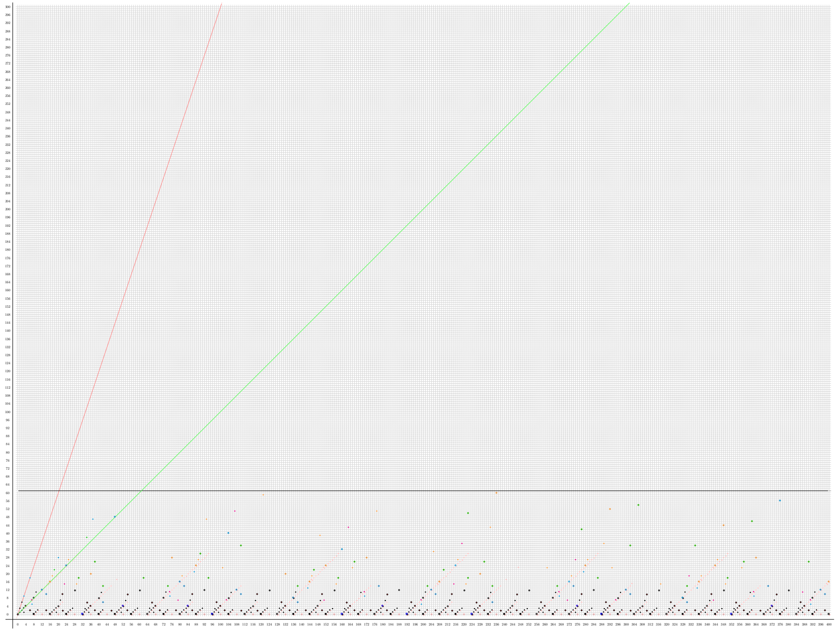

0 4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80 84 88 92 96 100 104 108 112 116 120 124 128 132 136 140 144 148 152 156 160 164 168 172 176 180 184 188 192 196 200 204 208 212 216 220 224 228 232 236 240 244 248 252 256 260 264 268 272 276 280 284 288 292 296 300 304 308 312 316 320 324 328 332 336 340 344 348 352 356 360 364 368 372 376 380 384 388 392 396 400

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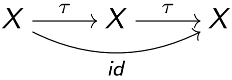

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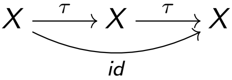

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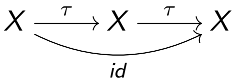

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- $K_{\mathbb{R}}(X)$: Grothendieck's construction

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- Localize at the prime 2,
  $MU_{\mathbb{R}}$ splits as a wedge of suspensions $BP_{\mathbb{R}}$

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- Mackey functors form an Abelian category!

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- Both at the same time: $\underline{\pi}_\bigstar(X)$

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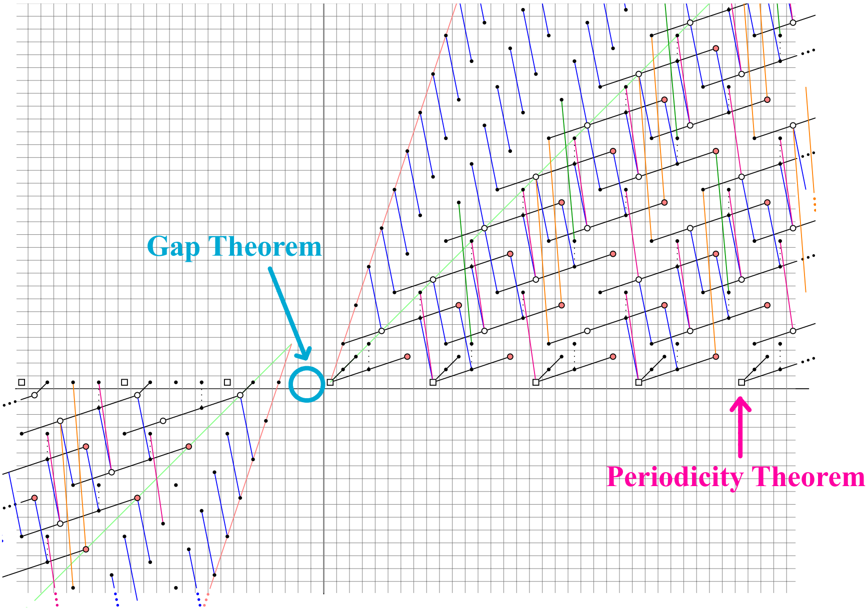
- Gap Theorem:
  $\pi_i\Omega = 0$ for $i = -1, -2, -3$.

Baby Ω



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$$BP^{((C_{2^m}))} \to \cdots \to BP^{((C_{2^m}))}\langle 2\rangle \to BP^{((C_{2^m}))}\langle 1\rangle \to BP^{((C_{2^m}))}\langle 0\rangle$$

- $\pi_*^u BP^{((C_{2^m}))}\langle n\rangle = \mathbb{Z}_{(2)}[C_{2^m}\cdot r_1, C_{2^m}\cdot r_3, \ldots, C_{2^m}\cdot r_{2^n-1}]$
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- $(BP^{((C_8))}\langle 1\rangle)^{C_8}$ also detects θ_j!

Slice Cells and the Slice Tower

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- $\mathcal{S}_{\leq n} = \{X |\ Map_G(Y, X) \simeq *, Y \in \mathcal{S}_{>n}\}$
- $P_{n+1}X \longrightarrow X \longrightarrow P^n X$
  $P_{n+1}X \in \mathcal{S}_{>n},\ P^n X \in \mathcal{S}_{\leq n}$

Slice tower:

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\begin{array}{ccccccccc}
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This is a spectral sequence of Mackey functors.

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It is also a Mackey functor of spectral sequences!

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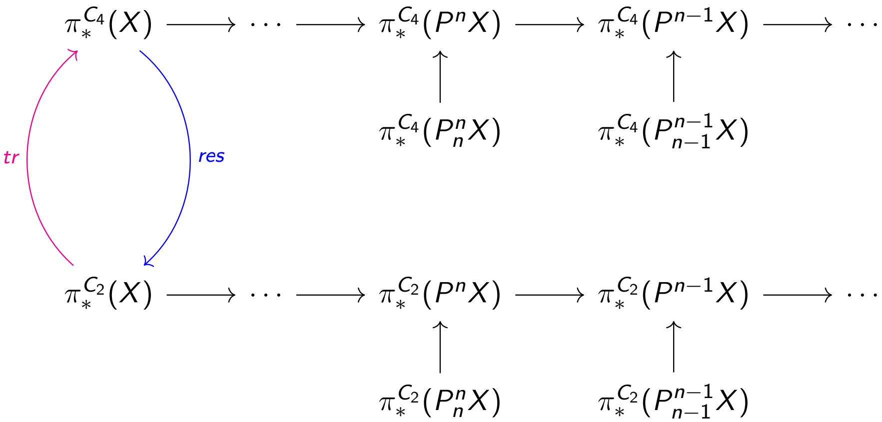

π_*^{C_4}(X) ⟶ ··· ⟶ π_*^{C_4}(P^n X) ⟶ π_*^{C_4}(P^{n-1} X) ⟶ ···
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tr
res
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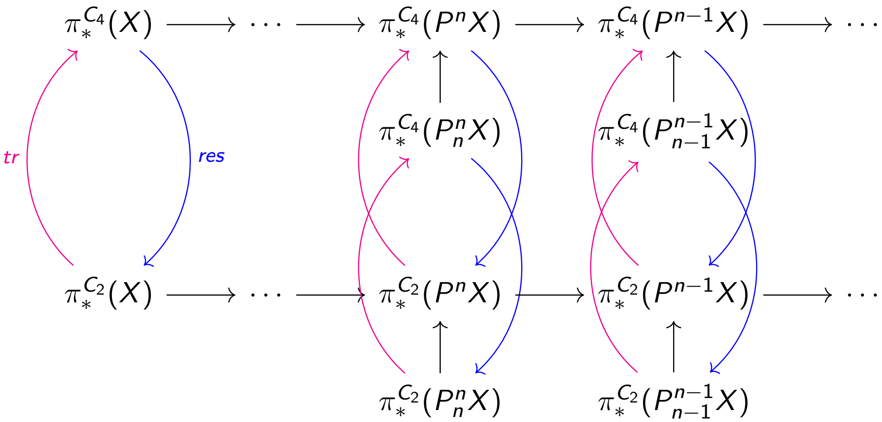

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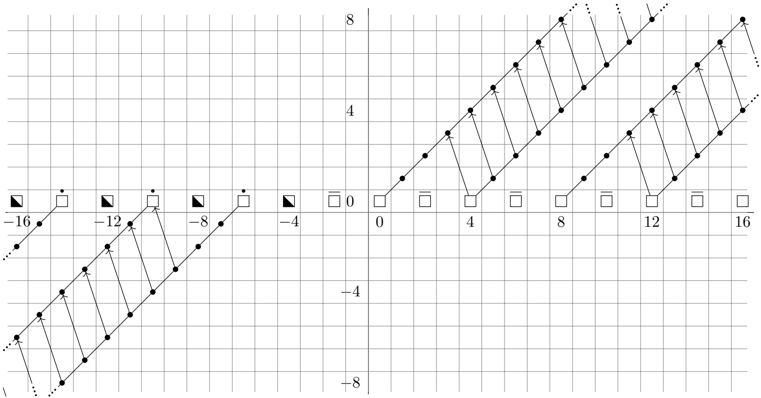
C_2-SliceSS($K_{\mathbb{R}}$): $\underline{\pi}_* K_{\mathbb{R}}$



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$$\begin{aligned} &\pi^{C_2}_{\star}(S^{n\rho} \wedge H\underline{\mathbb{Z}}) \\ &= H^{C_2}_{\star}(S^{n\rho}; \underline{\mathbb{Z}}) \\ &= \pi^{C_2}_{\star - n\rho} H\underline{\mathbb{Z}} \end{aligned}$$

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- Note that $RO(C_2) = \mathbb{Z} \oplus \mathbb{Z}$ generated by 1 and σ

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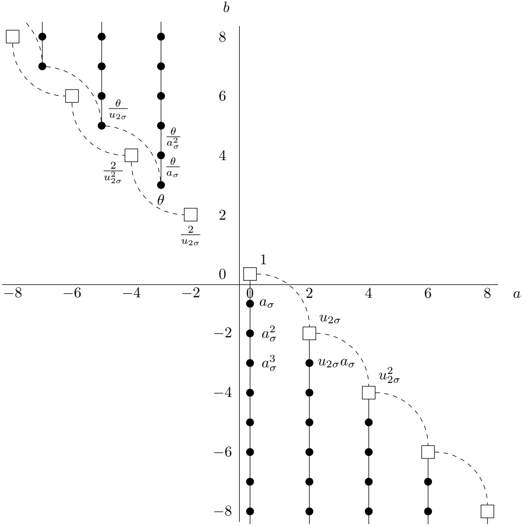
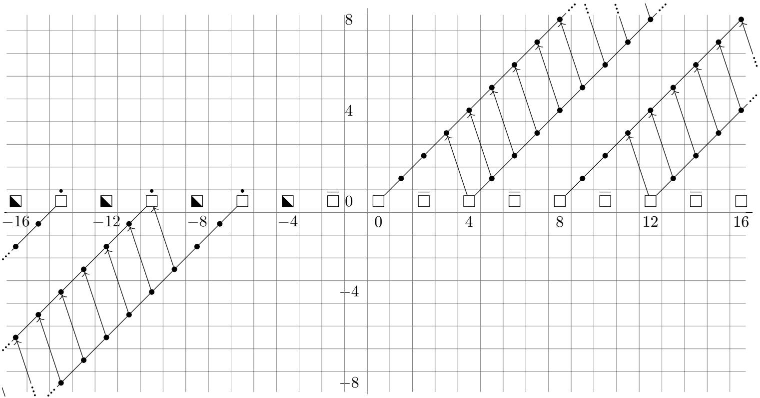
- $a_\sigma : S^0 \longrightarrow S^\sigma$
- $u_{2\sigma}$: generator of $H^{C_2}_2(S^{2\sigma}; \underline{\mathbb{Z}}) = \mathbb{Z}$

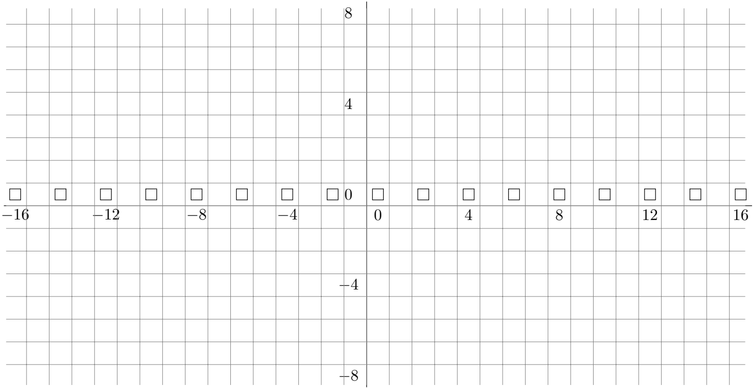
$\pi^{C_2}_{a+b\sigma} H\underline{\mathbb{Z}}$



C_2-SliceSS($K_{\mathbb{R}}$): $\underline{\pi}_* K_{\mathbb{R}}$



C_2-SliceSS($K_{\mathbb{R}}$): $\pi^u_* K_{\mathbb{R}} = \pi_* KU$



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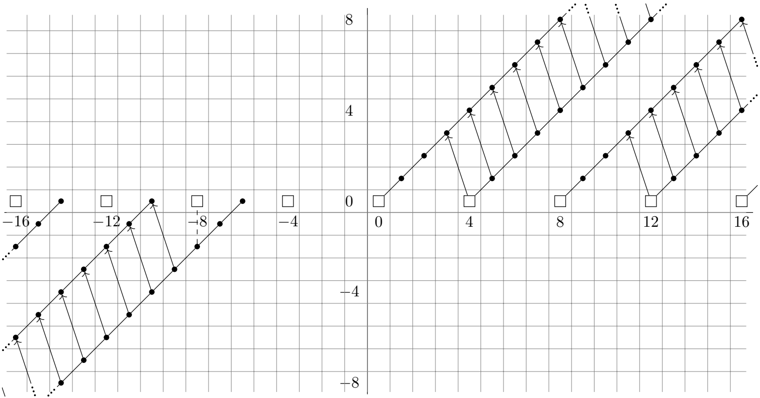

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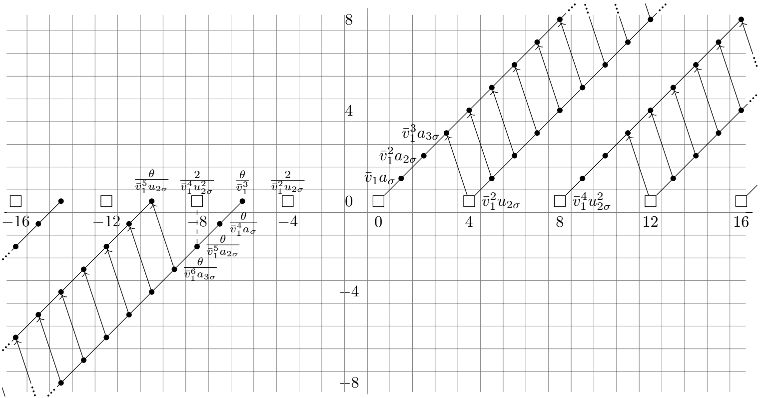

Two periodicities

In the $RO(C_2)$-grading:

- $\bar{v}_1 \in \pi_\rho^{C_2} K_\mathbb{R}$ gives ρ-periodicity

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Two equivalences:

$$S^{\rho} \wedge K_{\mathbb{R}} \simeq K_{\mathbb{R}}$$

$$S^{8} \wedge K_{\mathbb{R}} \simeq K_{\mathbb{R}}$$

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- What about Detection Theorems, Periodicity Theorems, and Gap Theorems?

Hurewicz Image of $BP_{\mathbb{R}}^{C_2}$

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What are the Hurewicz images?

Theorem (Li–S.–Wang–Xu)

The Hopf, Kervaire, and $\bar{\kappa}$-families are detected by the Hurewicz map $\pi_* \mathbb{S} \to \pi_* BP_{\mathbb{R}}^{C_2}$.

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The C_2-fixed points of $BP_{\mathbb{R}}\langle n\rangle$ detects the first n elements of the Hopf- and Kervaire-family, and the first $(n-1)$ elements of the $\bar{\kappa}$-family.

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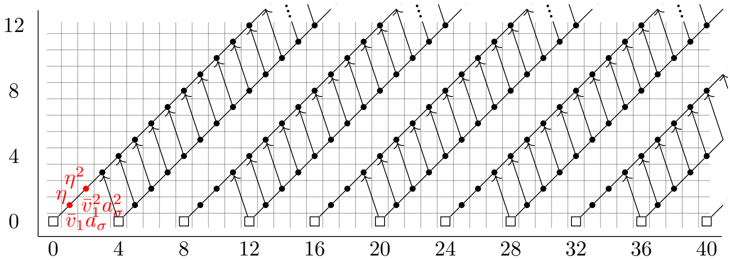
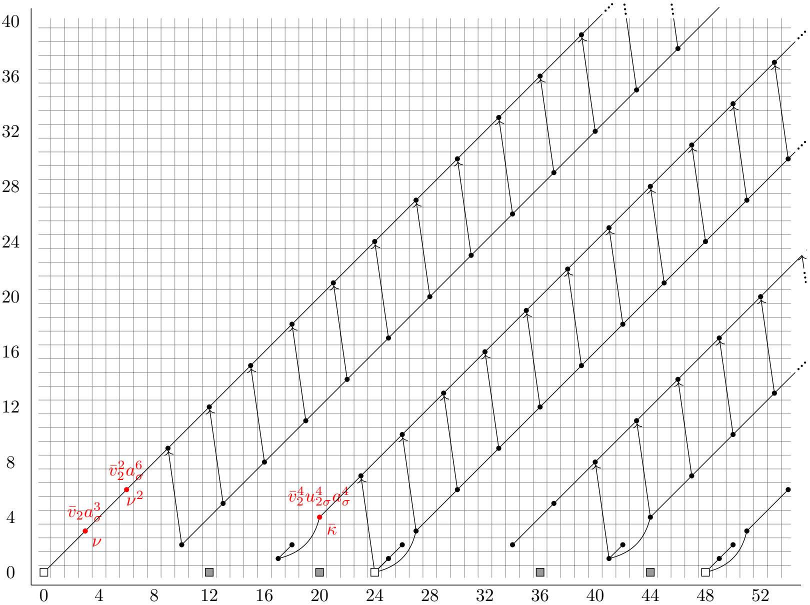
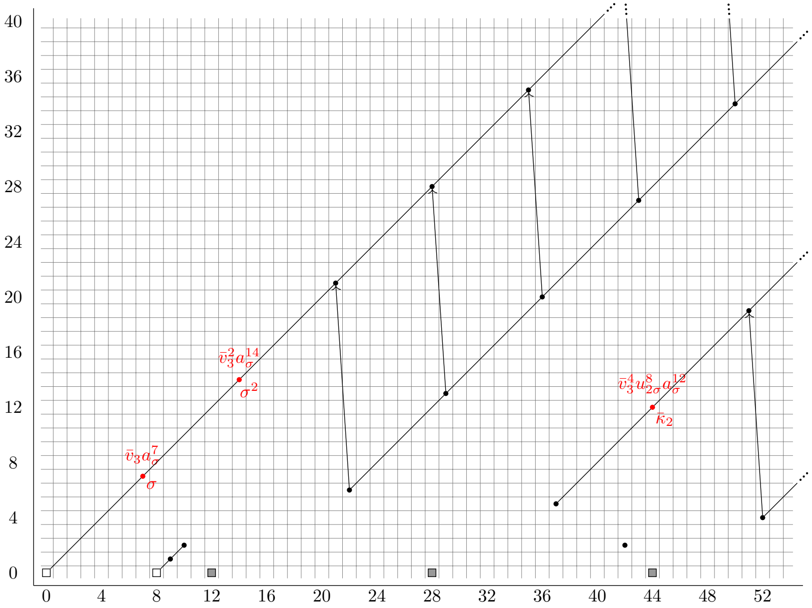


12
8
4
0
0
4
8
12
16
20
24
28
32
36
40
η^2
η
$\bar{v}_1^2 a_\sigma^2$
$\bar{v}_1 a_\sigma$



$\bar{v}_2 a_\sigma^3$
ν
$\bar{v}_2^2 a_\sigma^6$
ν^2
$\bar{v}_2^4 u_{2\sigma}^4 a_\sigma^4$
$\bar{\kappa}$
0
4
8
12
16
20
24
28
32
36
40
44
48
52



$\bar{v}_3 a_\sigma^7$
σ
$\bar{v}_3^2 a_\sigma^{14}$
σ^2
$\bar{v}_3^4 u_{2\sigma}^8 a_\sigma^{12}$
$\bar{\kappa}_2$

Real Orientation

central $C_2 \subset \mathbb{S}_n$: acts on E_{n*} by $[-1]$.

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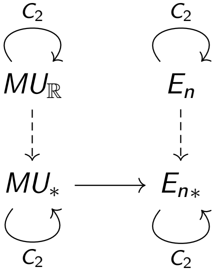

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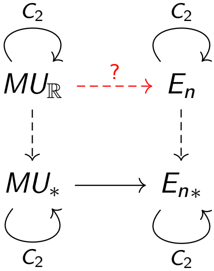

Can we lift it?

Theorem (Hahn–S.)

The Morava E-theory is Real oriented: it receives a C_2-equivariant map

$$MU_{\mathbb{R}} \longrightarrow E_n$$

from the Real bordism spectrum $MU_{\mathbb{R}}$.

$$
\begin{array}{ccccccccc}
BP_{\mathbb{R}} & \longrightarrow & \cdots & \longrightarrow & BP_{\mathbb{R}}\langle 3\rangle & \longrightarrow & BP_{\mathbb{R}}\langle 2\rangle & \longrightarrow & BP_{\mathbb{R}}\langle 1\rangle \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & E_3 & & E_2 & & E_1 \\
 & & & & \circlearrowleft C_2 & & \circlearrowleft C_2 & & \circlearrowleft C_2
\end{array}
$$

$$\begin{array}{ccccccc} BP_{\mathbb{R}}^{C_2} \longrightarrow \cdots \longrightarrow & BP_{\mathbb{R}}\langle 3\rangle^{C_2} & \longrightarrow & BP_{\mathbb{R}}\langle 2\rangle^{C_2} & \longrightarrow & BP_{\mathbb{R}}\langle 1\rangle^{C_2} \\ & \downarrow & & \downarrow & & \downarrow \\ & E_3^{hC_2} & & E_2^{hC_2} & & E_1^{hC_2} \end{array}$$

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Detection theorems for $BP_{\mathbb{R}}\langle n\rangle^{C_2}$
$\Longrightarrow$ Detection theorems for $E_n^{hC_2}$.

Theorem (Li–S.–Wang–Xu, Hahn–S.)

$E_n^{hC_2}$ detects the first n elements of the Hopf- and Kervaire-family, and the first $(n-1)$ elements of the $\bar{\kappa}$-family.

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The Hurewicz images of $E_n^{hC_2}$ and $BP_{\mathbb{R}}\langle n\rangle^{C_2}$ are the same.

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What about bigger groups?

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Theorem (Hahn–Shi)

Let $G \subset \mathbb{S}_n$ be a finite subgroup containing the central subgroup C_2. There is a G-equivariant map

$$MU^{((G))} \longrightarrow E_n.$$

For $G = C_4$:

$$\begin{array}{ccccccc}
BP^{((C_4))} & \longrightarrow & BP^{((C_4))}\langle 3\rangle & \longrightarrow & BP^{((C_4))}\langle 2\rangle & \longrightarrow & BP^{((C_4))}\langle 1\rangle \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & E_6 & & E_4 & & E_2 \\
 & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\
 & & C_4 & & C_4 & & C_4
\end{array}$$

For $G = C_{2^m}$:

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 & & \downarrow & & \downarrow & & \downarrow \\
 & & E_{3\cdot 2^{m-1}} & & E_{2\cdot 2^{m-1}} & & E_{2^{m-1}} \\
 & & \circlearrowright & & \circlearrowright & & \circlearrowright \\
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Techniques developed by Hill, Hopkins, and Ravenel show that

- $\pi_* E_{n\cdot 2^{m-1}}^{hC_{2^m}}$ is periodic with period $2^{n\cdot 2^{m-1}+m+1}$.

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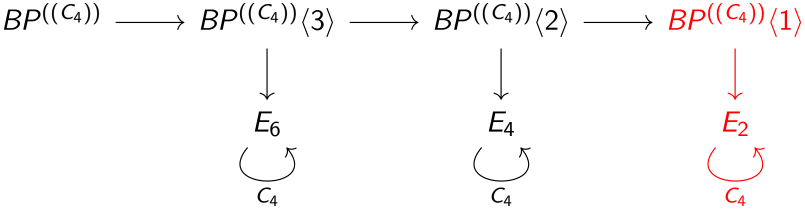
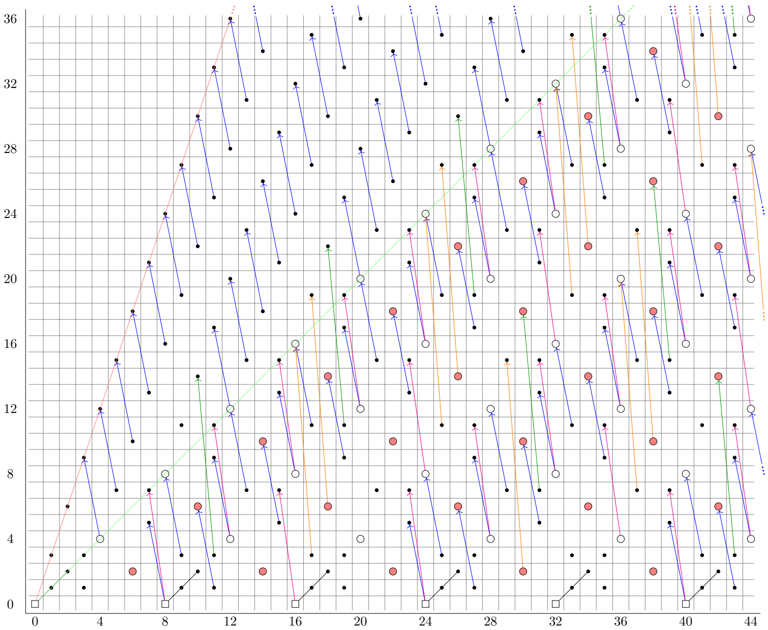
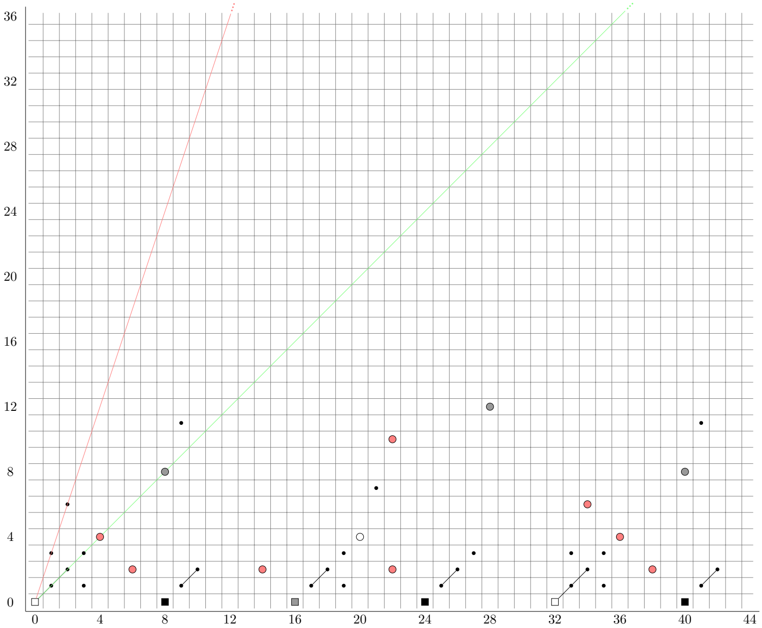


$BP^{((C_4))}$
$BP^{((C_4))}\langle 3\rangle$
$BP^{((C_4))}\langle 2\rangle$
$BP^{((C_4))}\langle 1\rangle$
E_6
E_4
E_2
C_4
C_4
C_4

SliceSS$(BP^{((C_4))}\langle 1\rangle)$



SliceSS($BP^{((C_4))}\langle 1\rangle$)



Three periodicities

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  - $S^{4-4\sigma} \wedge D^{-1}BP^{((C_4))}\langle 1 \rangle \simeq D^{-1}BP^{((C_4))}\langle 1 \rangle$
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  - $S^{8+8\sigma-8\lambda} \wedge D^{-1}BP^{((C_4))}\langle 1 \rangle \simeq D^{-1}BP^{((C_4))}\langle 1 \rangle$
- Together, they imply the 32-periodicity!
  $8\rho_4 + 4(4 - 4\sigma) + (8 + 8\sigma - 8\lambda)$
  $= 8(1 + \sigma + \lambda) + 4(4 - 4\sigma) + (8 + 8\sigma - 8\lambda)$
  $= 32$

Hurewicz images

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Hurewicz images

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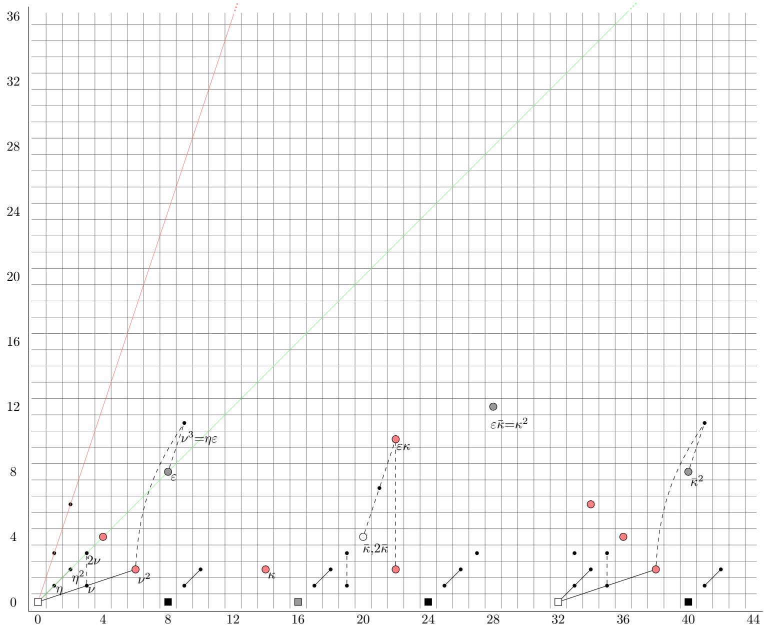
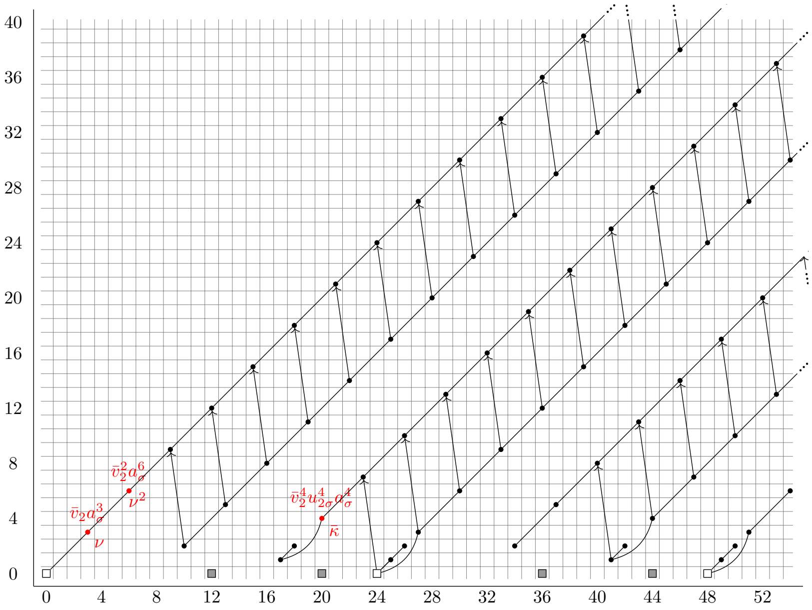
$\Longrightarrow$ These elements are also detected in $\pi^{C_4}_* BP^{((C_4))}\langle 1 \rangle$

SliceSS($BP^{((C_4))}\langle 1\rangle$)



SliceSS($BP_{\mathbb{R}}\langle 2\rangle$)



Stabilization of Filtration

For ν:

$\pi_*^{C_{2^m}} BP^{((C_{2^m}))}$	C_2	C_4	C_8	C_{16}
Filtration	3	1	1	1
Order	2	4	4	4

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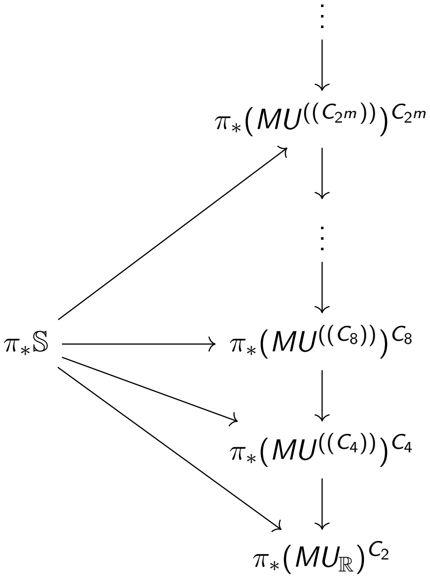
For θ_n:

$\pi_*^{C_{2^m}} BP^{((C_{2^m}))}$	C_2	C_4	C_8	C_{16}
η^2	2	2	2	2
ν^2	6	2	2	2
σ^2	14	10	2	2
θ_4	30	18	2	2

Hill's Detection Tower



Conjecture (Hill)

1. *As m increases, the filtration of a spherical class detected in* $\pi_*(MU^{((C_{2^m}))})^{C_{2^m}}$ *decreases and eventually stabilizes to its Adams–Novikov filtration.*

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2. *When this class first moves into its stable filtration, it achieves its maximal order that is detected by $\pi_*(MU^{((C_{2^m}))})^{C_{2^m}}$ for all m.*

Conjecture (Hill)

1. *As m increases, the filtration of a spherical class detected in* $\pi_*(MU^{((C_{2^m}))})^{C_{2^m}}$ *decreases and eventually stabilizes to its Adams–Novikov filtration.*
2. *When this class first moves into its stable filtration, it achieves its maximal order that is detected by* $\pi_*(MU^{((C_{2^m}))})^{C_{2^m}}$ *for all m.*

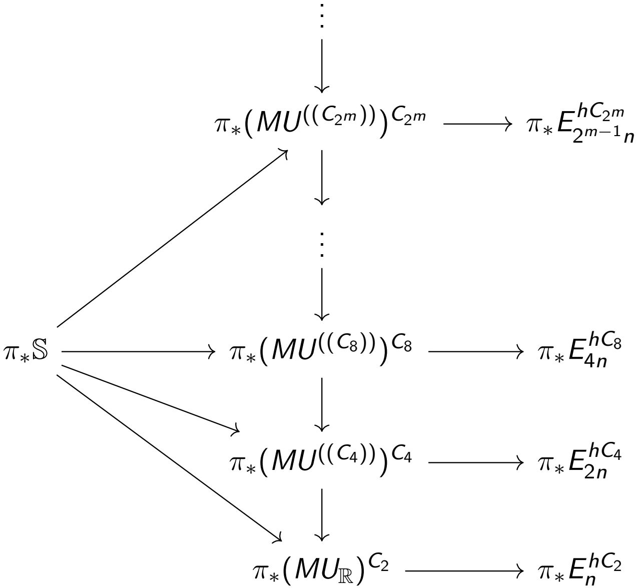
Question (Hill)

What is the Hurewicz image of $\varprojlim \pi_*(MU^{((C_{2^m}))})^{C_{2^m}}$*?*

Hill's Detection Tower

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
 & \nearrow & \pi_*(MU^{((C_{2^m}))})^{C_{2^m}} \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
\pi_*\mathbb{S} & \longrightarrow & \pi_*(MU^{((C_8))})^{C_8} \\
 & \searrow & \downarrow \\
 & & \pi_*(MU^{((C_4))})^{C_4} \\
 & \searrow & \downarrow \\
 & & \pi_*(MU_{\mathbb{R}})^{C_2}
\end{array}
$$

Hill's Detection Tower



Slice SS vs. Homotopy Fixed Point SS

$$\begin{array}{ccc} \mathrm{SliceSS}(X) & \longrightarrow & \mathrm{HFPSS}(X) \\ \Downarrow & & \Downarrow \\ \pi_* X^G & \longrightarrow & \pi_* X^{hG} \end{array}$$

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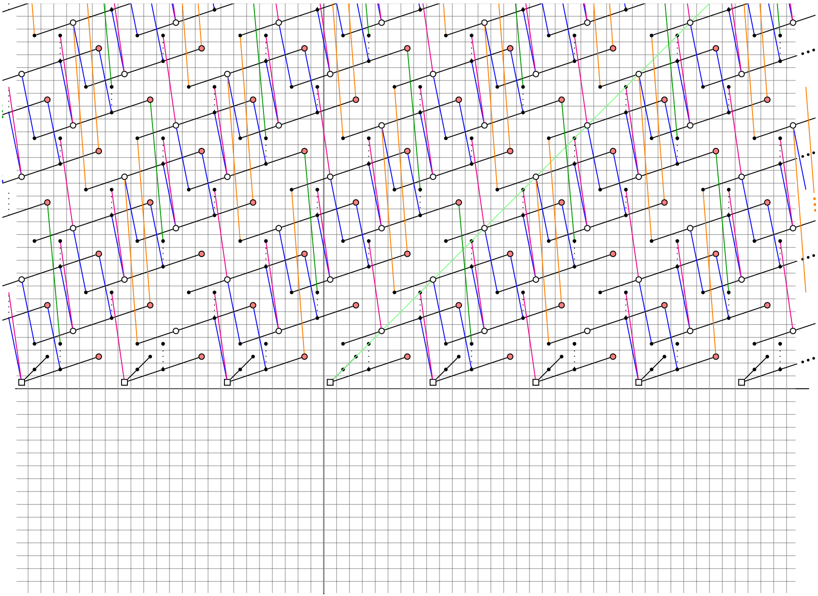
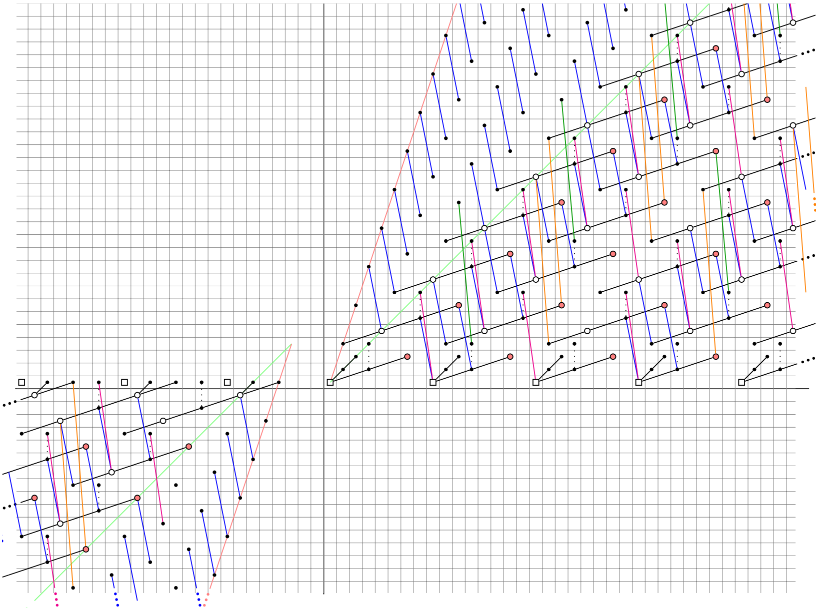
This is an isomorphism under the line of slope 1

HFPSS($E_2^{hC_4}$)



SliceSS($E_2^{hC_4}$)



Advantages of the Slice SS

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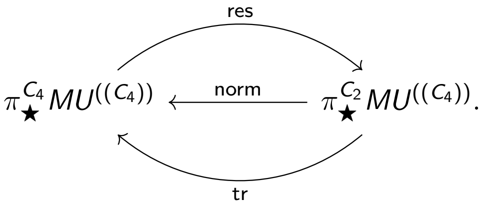
- These differentials imply the Periodicity Theorem.
- Slice SS + isomorphism + Periodicity Theorem $\overset{\text{recovers}}{\Longrightarrow}$ HFPSS

Computing Differentials

$MU^{((C_4))}$: commutative C_4-spectrum



Computing Differentials

$$C_4\text{-}\,\mathrm{SliceSS}(MU^{((C_4))}) \underset{\mathrm{tr}}{\overset{\mathrm{res}}{\rightleftarrows}} C_2\text{-}\,\mathrm{SliceSS}(MU^{((C_4))}).$$

$$C_4\text{-}\,\mathrm{SliceSS}(MU^{((C_4))}) \overset{\mathit{norm}}{\dashleftarrow} C_2\text{-}\,\mathrm{SliceSS}(MU^{((C_4))})$$

Computing Differentials

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- Restriction and transfer are maps of spectral sequences.

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$$C_4\text{-}\,\mathrm{SliceSS}(MU^{((C_4))}) \xleftarrow{\;norm\;} C_2\text{-}\,\mathrm{SliceSS}(MU^{((C_4))})$$

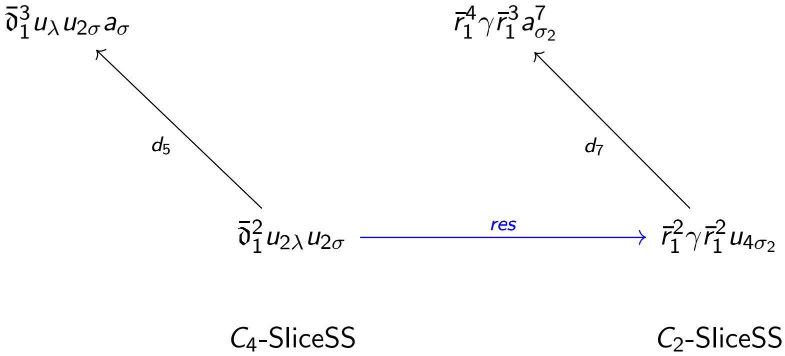
- Restriction and transfer are maps of spectral sequences.
- Norm “stretches out” differentials.

Theorem (Hill–Hopkins–Ravenel)

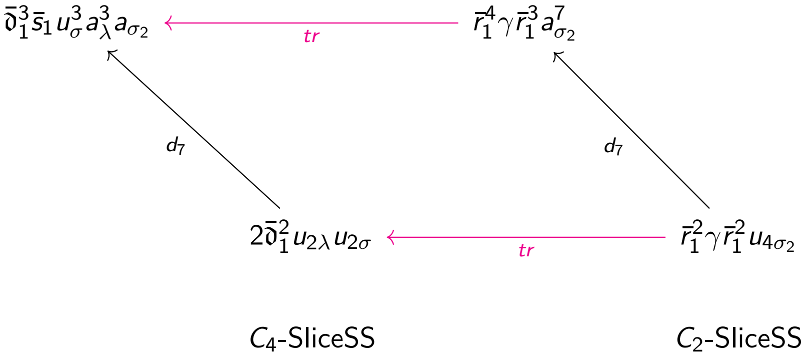
Let $d_r(x) = y$ be a d_r-differential in C_2-SliceSS$(MU^{((C_4))})$. If both $a_\sigma N_{C_2}^{C_4} x$ and $N_{C_2}^{C_4} y$ survive to the E_{2r-1}-page in C_4-SliceSS$(MU^{((C_4))})$, then

$$d_{2r-1}(a_\sigma N_{C_2}^{C_4} x) = N_{C_2}^{C_4} y.$$

Restriction



Transfer

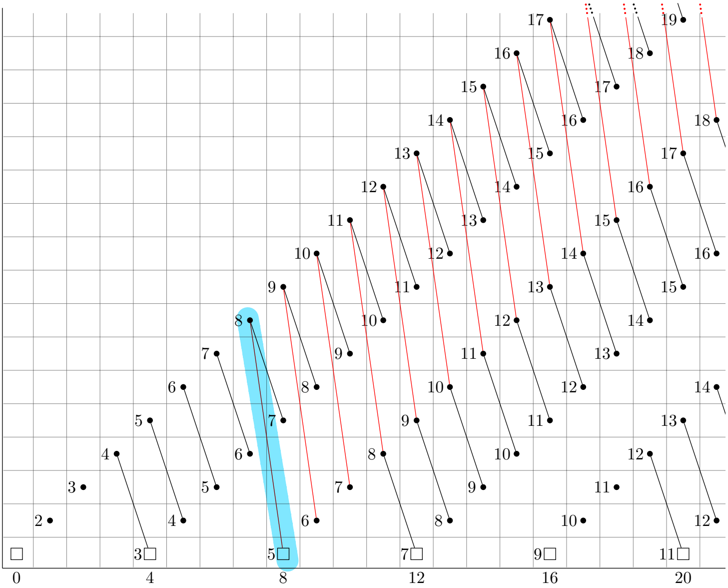

$\bar{\mathfrak{d}}_1^3 \bar{s}_1 u_\sigma^3 a_\lambda^3 a_{\sigma_2}$
tr
$\bar{r}_1^4 \gamma \bar{r}_1^3 a_{\sigma_2}^7$
d_7
d_7
$2\bar{\mathfrak{d}}_1^2 u_{2\lambda} u_{2\sigma}$
tr
$\bar{r}_1^2 \gamma \bar{r}_1^2 u_{4\sigma_2}$
C_4-SliceSS
C_2-SliceSS

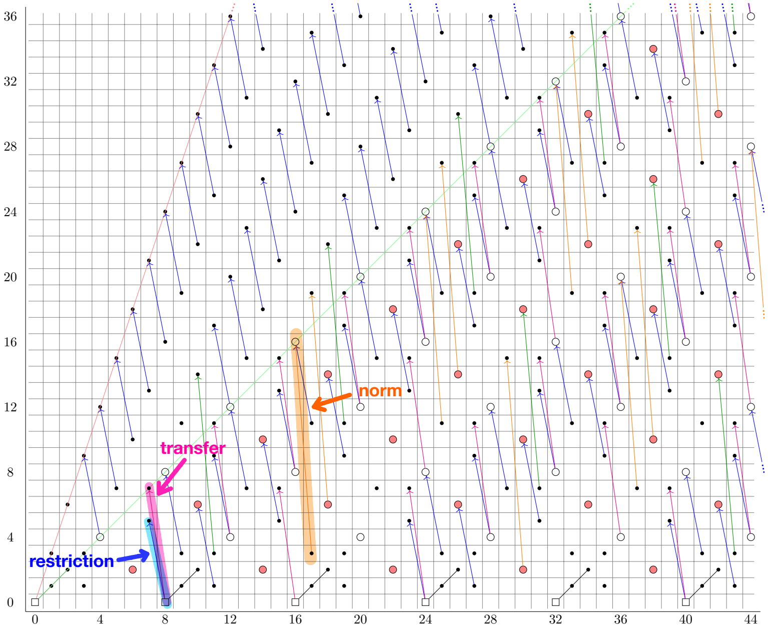
Norm

$$C_2\text{-SliceSS: } d_7(u_{4\sigma_2}) = \bar{r}_1^2 \gamma \bar{r}_1 a_{\sigma_2}^7$$

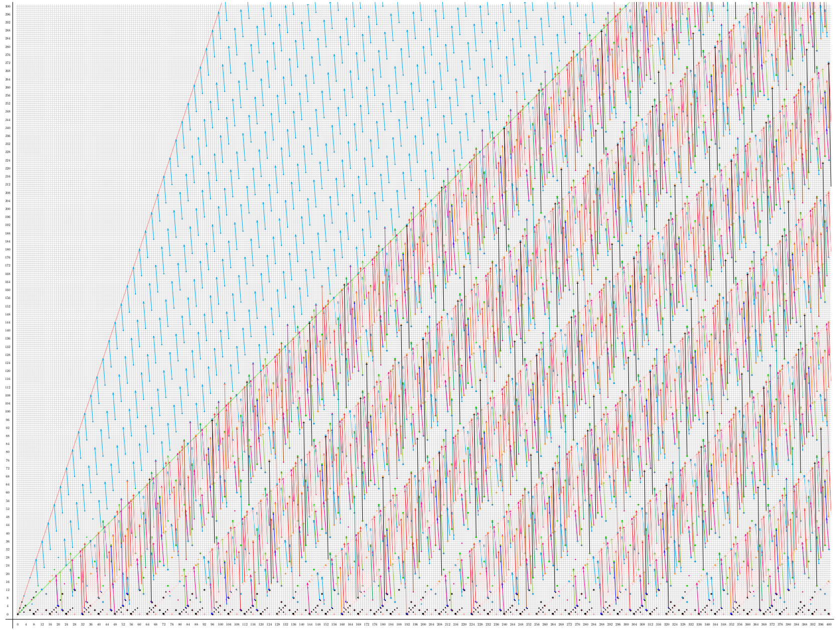
$$\Big\downarrow \text{ apply norm}$$

$$C_4\text{-SliceSS: } d_{13}(u_{4\lambda} a_\sigma) = \bar{\mathfrak{d}}_1^3 a_\lambda^7$$

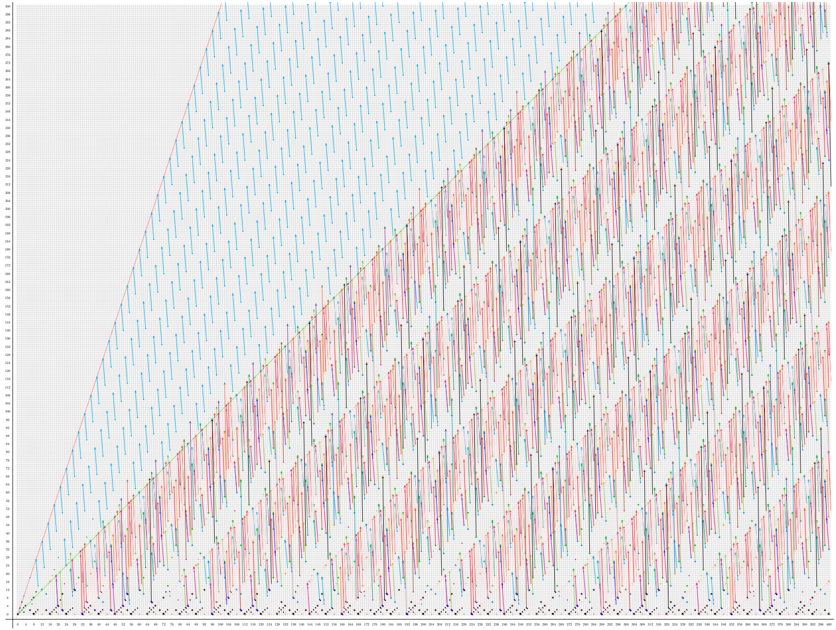

0
4
8
12
16
20


transfer
norm
restriction
0
4
8
12
16
20
24
28
32
36
40
44

?

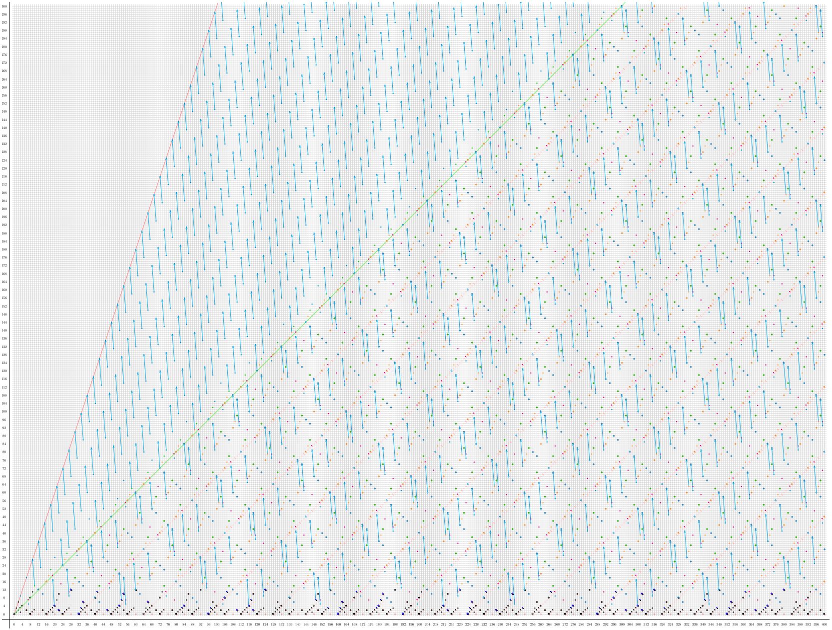


SliceSS($BP^{((C_4))}\langle 2\rangle$)

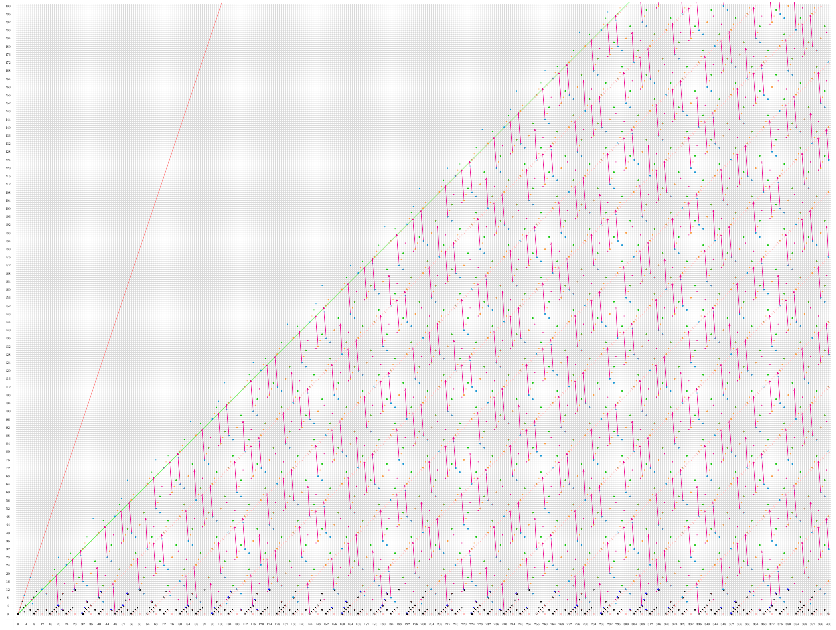


$$
\begin{array}{ccccccc}
BP^{((C_4))} & \longrightarrow & BP^{((C_4))}\langle 3\rangle & \longrightarrow & {\color{red}BP^{((C_4))}\langle 2\rangle} & \longrightarrow & BP^{((C_4))}\langle 1\rangle \\
 & & \downarrow & & {\color{red}\downarrow} & & \downarrow \\
 & & E_6 & & {\color{red}E_4} & & E_2 \\
 & & \circlearrowright C_4 & & {\color{red}\circlearrowright C_4} & & \circlearrowright C_4
\end{array}
$$

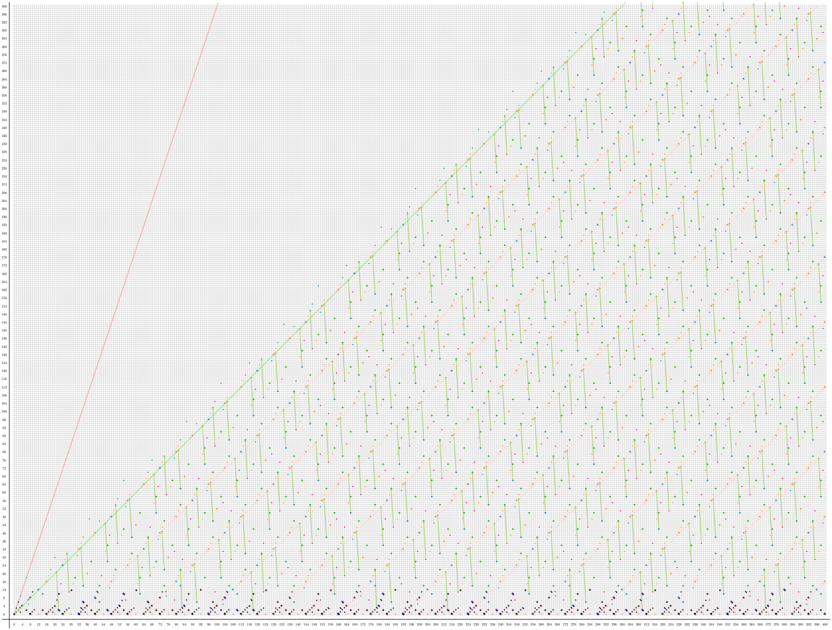
SliceSS($BP^{((C_4))}\langle 2 \rangle$) : d_{13}



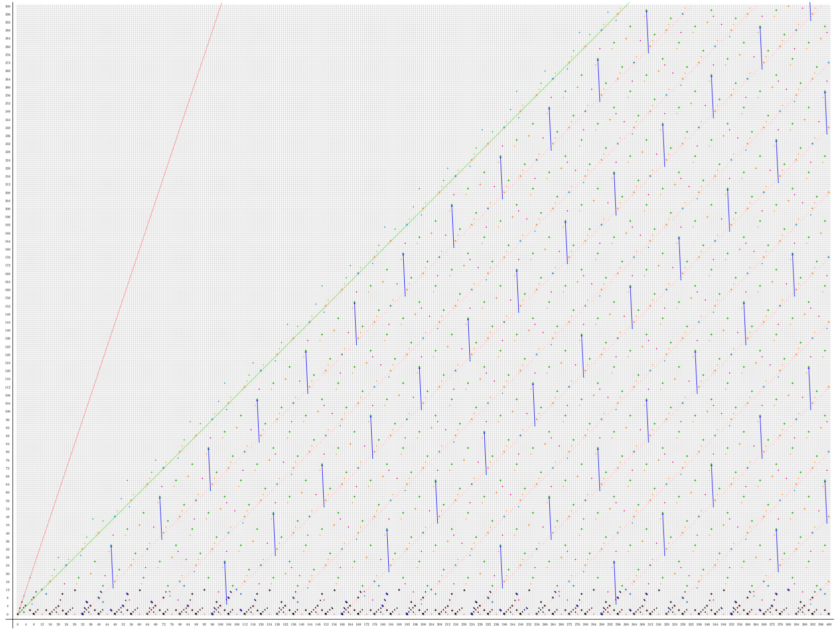
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{15}



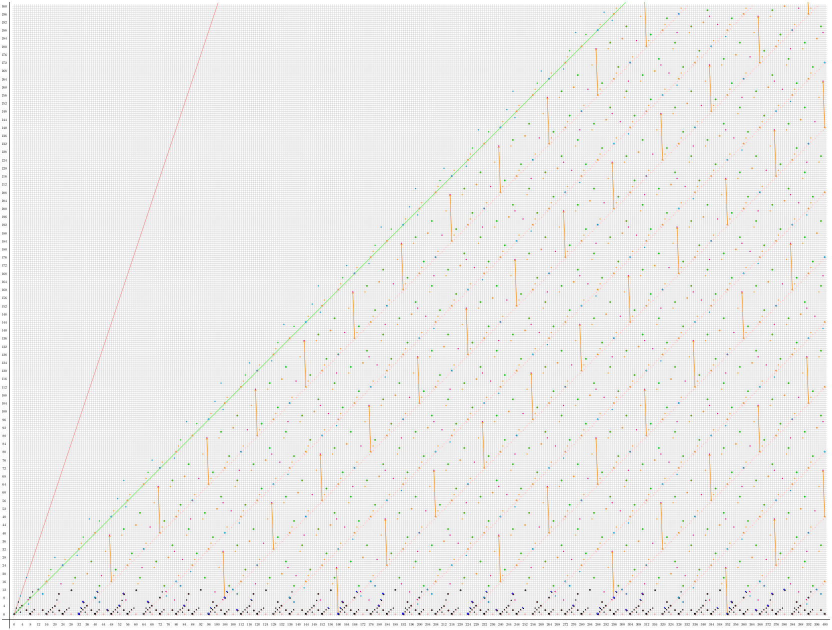
SliceSS$(BP^{((C_4))}\langle 2\rangle)$: d_{19}



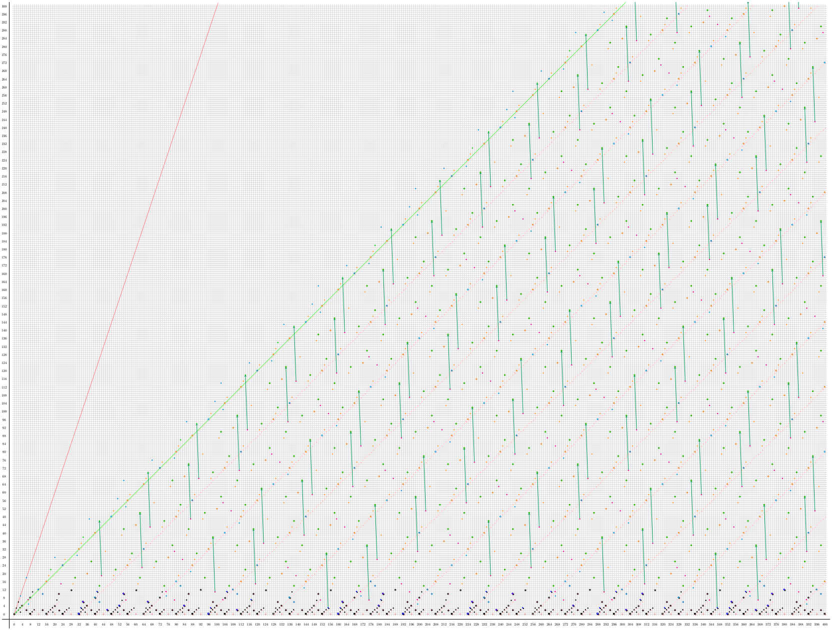
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{21}



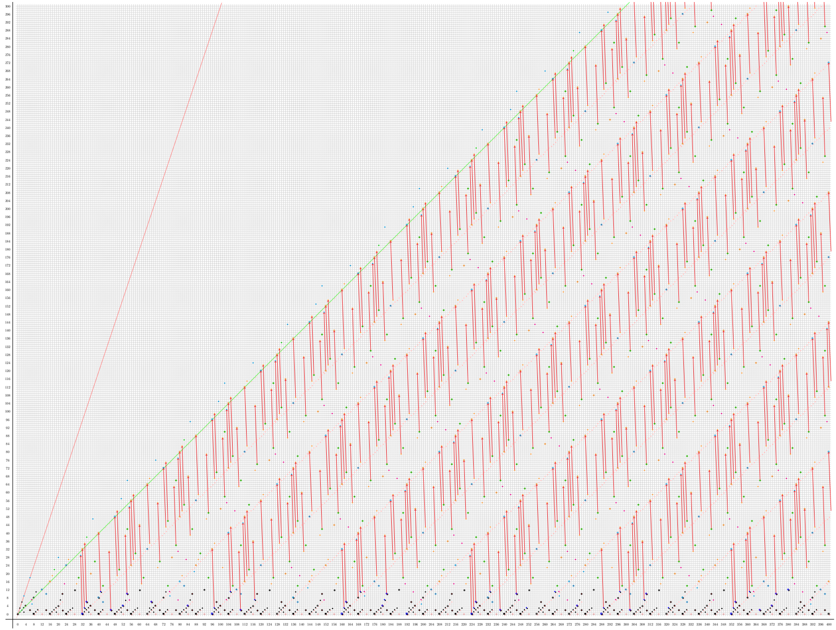
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{23}



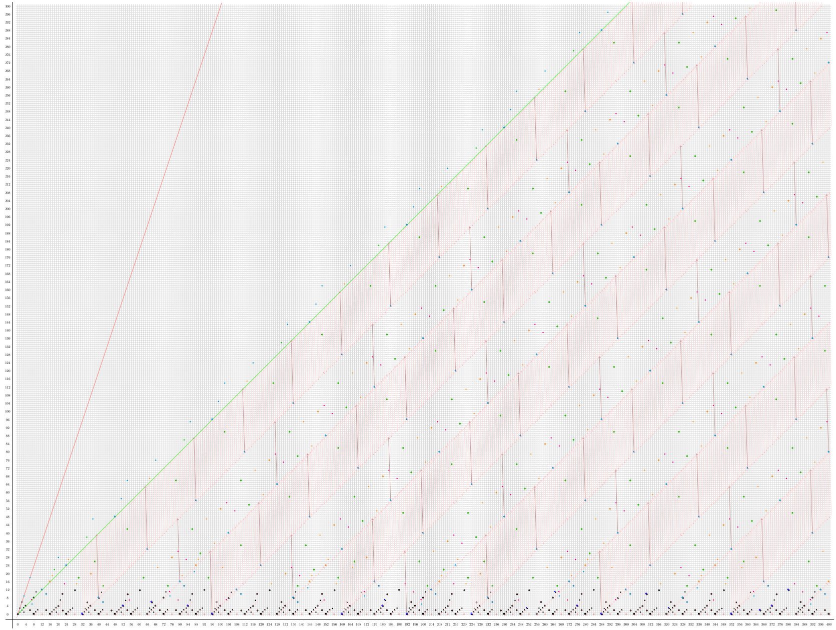
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{27}



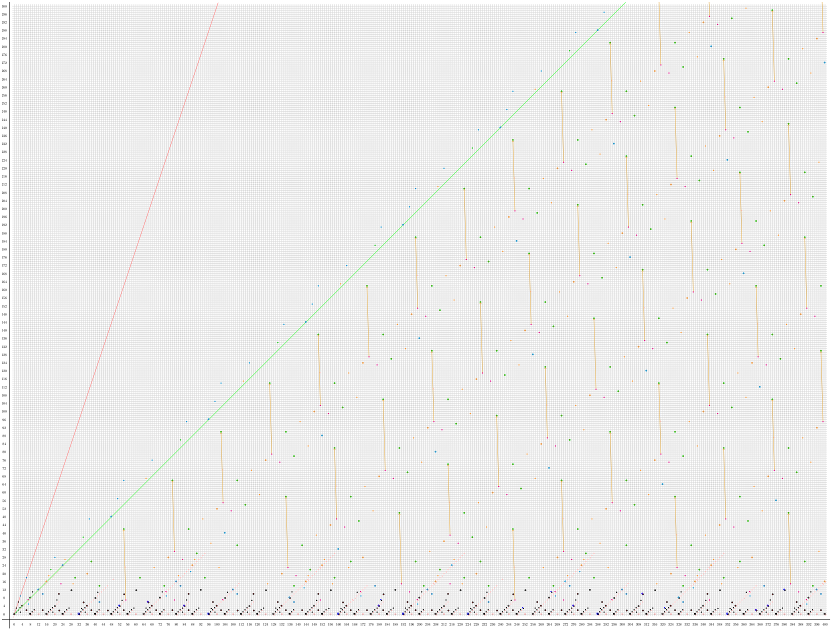
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{29}



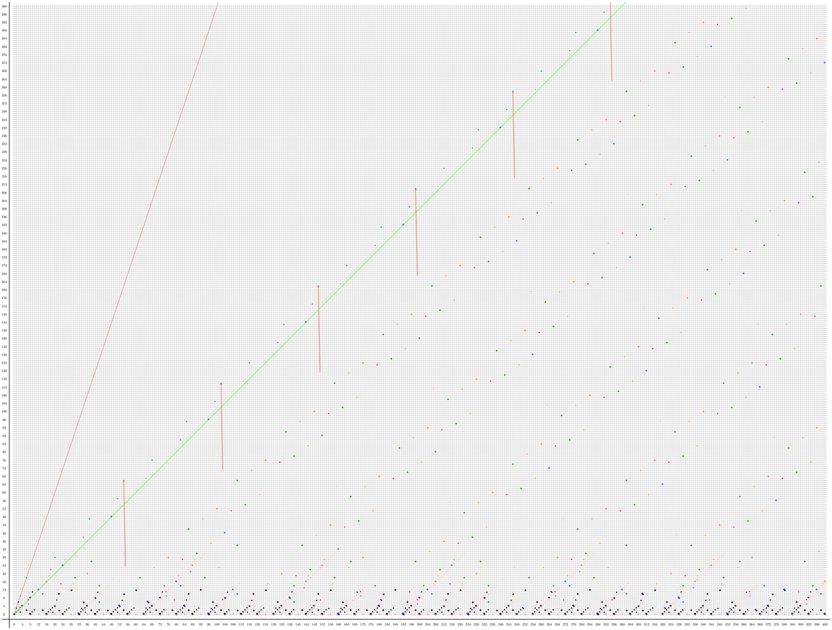
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{31}



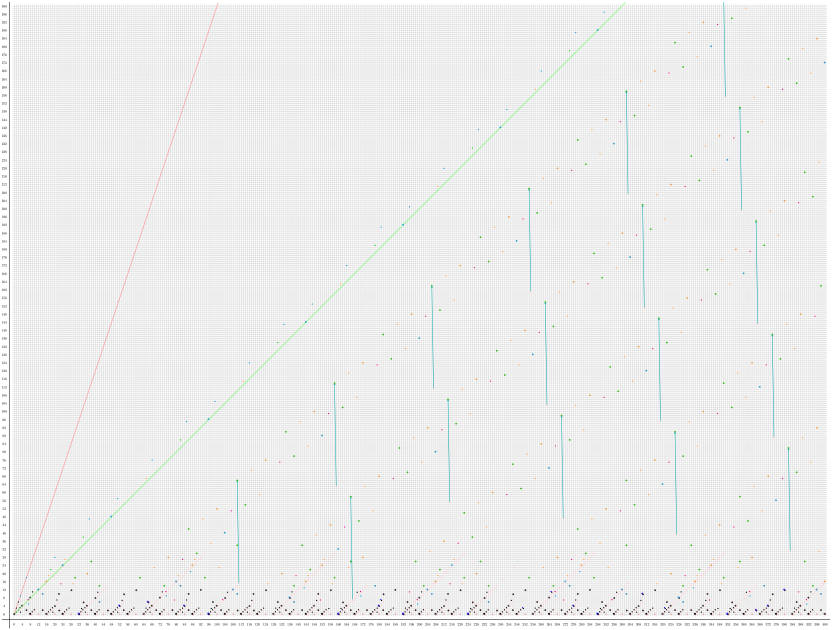
SliceSS$(BP^{((C_4))}\langle 2\rangle)$: d_{35}



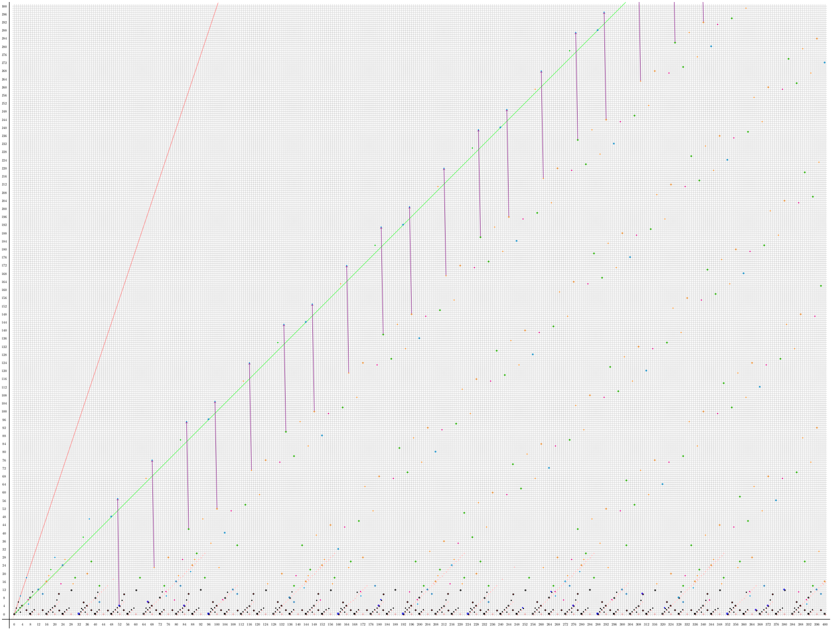
SliceSS$(BP^{((C_4))}\langle 2\rangle)$: d_{43}



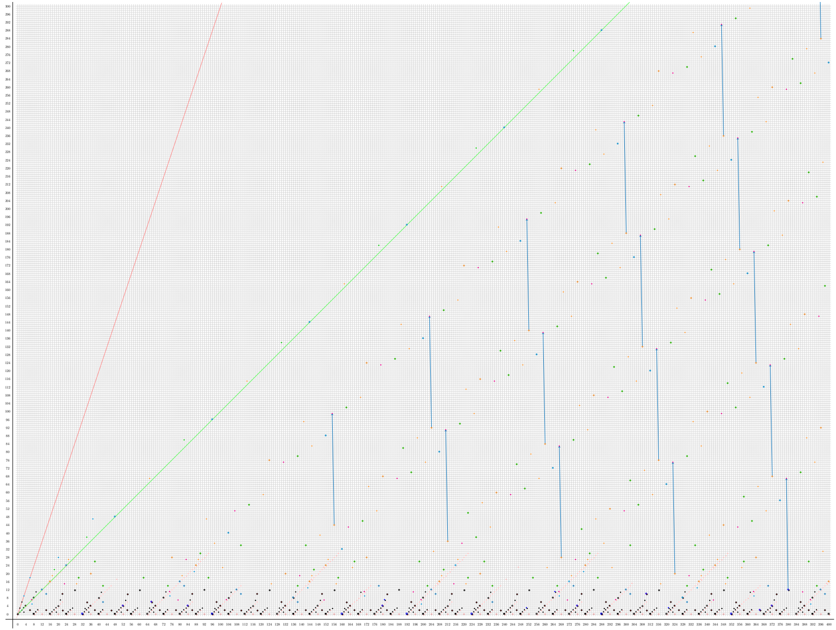
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{51}



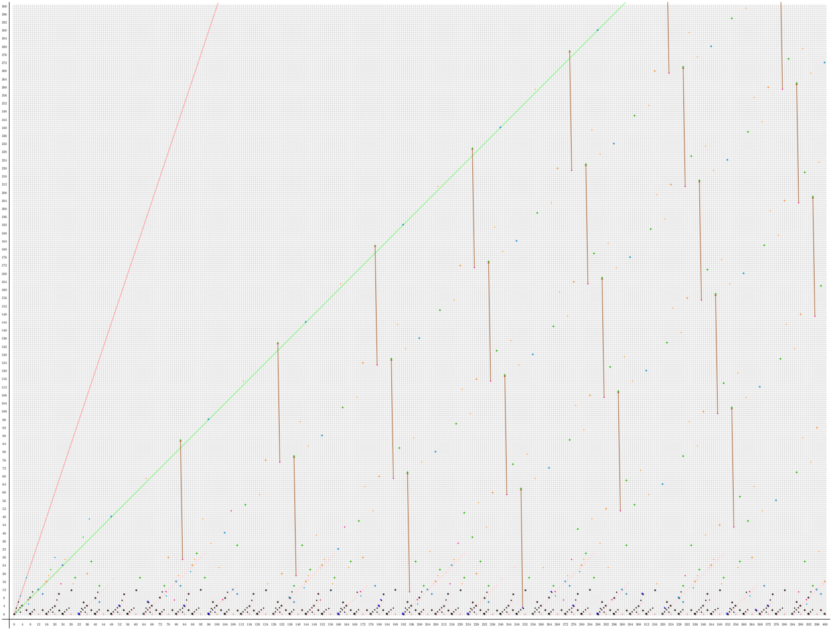
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{53}



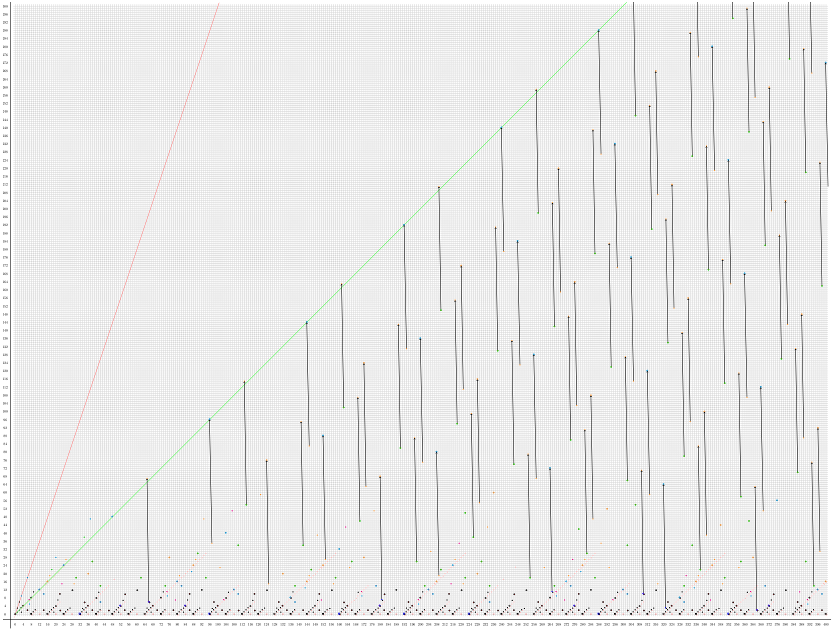
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{55}



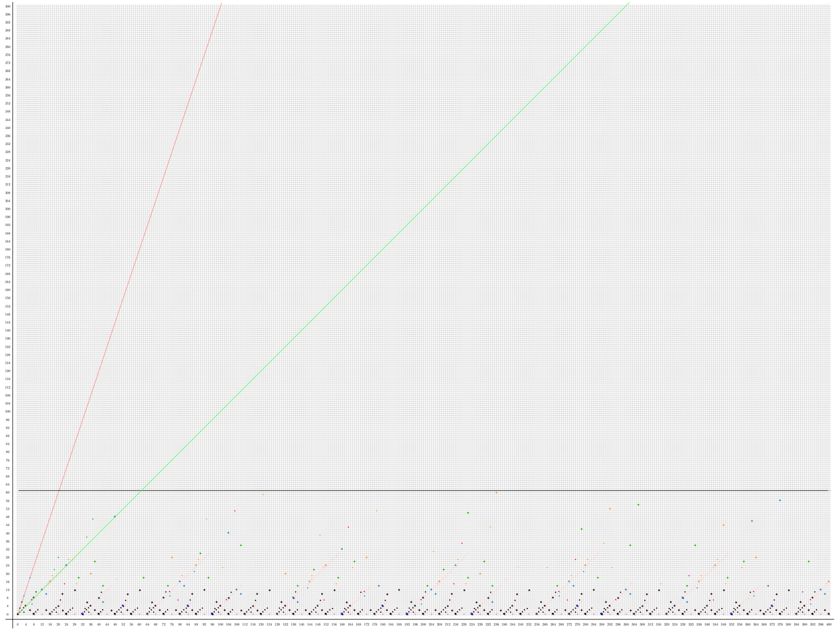
SliceSS($BP^{((C_4))}\langle 2\rangle$) : d_{59}



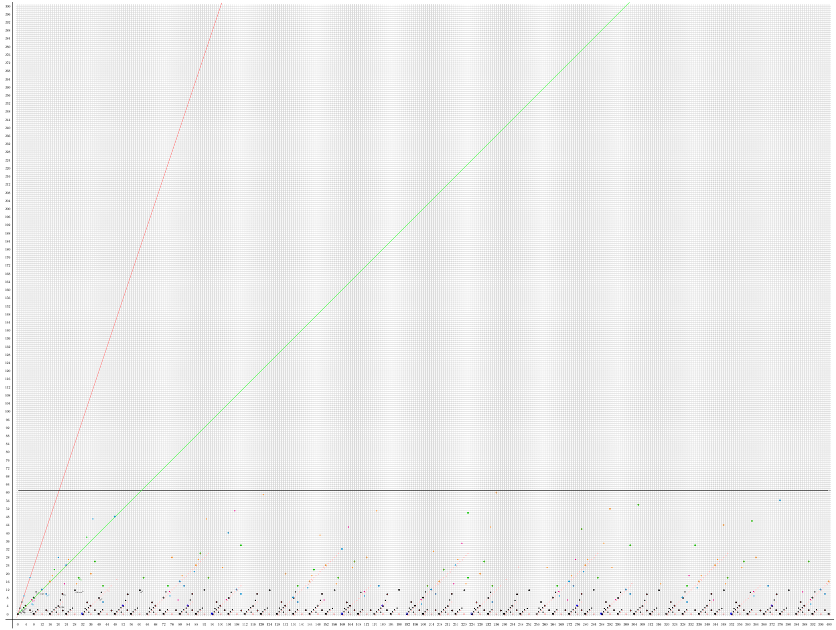
SliceSS$(BP^{((C_4))}\langle 2\rangle)$: d_{61}



SliceSS$(BP^{((C_4))}\langle 2 \rangle)$: E_∞



SliceSS($BP^{((C_4))}\langle 2\rangle$): Hurewicz Images



Periodicities

- There are three periodicities for $D^{-1}BP^{((C_4))}\langle 2\rangle$:

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- These periodicities imply that $D^{-1}BP^{((C_4))}\langle 2\rangle$ is 384-periodic!
  $32 \cdot (3\rho_4) + 3 \cdot (32 + 32\sigma - 32\lambda) + 24 \cdot (8 - 8\sigma)$
  $= 32 \cdot (3 + 3\sigma + 3\lambda) + 3 \cdot (32 + 32\sigma - 32\lambda) + 24 \cdot (8 - 8\sigma)$
  $= 384$

Thank you all for coming!