Simplicial James-Hopf map and decompositions of the unstable Adams spectral sequence for suspensions —Joint with Fedor Pavutnitskiy

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This is an introductory talk on the first part of Fedor Pavutnitskiy's PhD thesis.

Purpose of the project

- Towards to study the action of the Cohen group on the lower central series spectral sequences (LCSSS) converging to $\pi_*(\Omega\Sigma X)$ from Milnor's construction, which is part of Cohen's program towards to attacking the Barratt conjecture on the exponent problem.
- Decomposing LCSSS. For finite complexes X with two or more cell, the growth of the number of $\mathbb{Z}/p^r$-summands in $\pi_*(\Sigma X)$ goes exponentially in general if it occurs. The decompositions of spectral sequences help for controlling the differentials.
- The current work concludes that the Cohen group acts on the lower central series spectral sequences (LCSSS) converging to $\pi_*(\Omega\Sigma X)$. Further exploration may produce the (higher degree) operations on LCSSS.

Feature of Cohen groups

$$
\begin{array}{ccccc}
\mathrm{coalg}(T(-),T(-)) & \overset{\cong}{\twoheadleftarrow} & \mathfrak{H}_\infty & \longrightarrow & [\Omega\Sigma(-),\Omega\Sigma(-)] \\
\downarrow & & \downarrow\!\!\downarrow & & \downarrow \Omega \\
\prod_{n=1}^{\infty} \mathrm{Hom}(L_n(-),L_n(-)) & \longleftarrow & \mathfrak{R}_\infty & \longrightarrow & [\Omega^2\Sigma(-),\Omega^2\Sigma(-)]
\end{array}
$$

James Construction

Let X be a space with a basepoint $*$. The James construction $J(X)$ is the free monoid generated by X subject to the single relation that $* = 1$. More precisely,

- $J_n(X)$ is the quotient space of $X^{\times n}$ by the equivalence relations generated by $(x_1, \cdots, x_{i-1}, *, x_i, \cdots, x_{n-1}) \sim (x_1, \cdots, x_{j-1}, *, x_j, \cdots, x_{n-1})$.
- $J(X) = \bigcup\limits_n J_n(X)$ with weak topology.

James Theorem. If X is path-connected, then

$$J(X) \overset{w}{\simeq} \Omega\Sigma X,$$

i.e. $J(X)$ is (weakly) homotopy equivalent to the loop space of the suspension of X.

James-Hopf Maps

- From the definition, $J_n(X)/J_{n-1}(X) \cong X^{\wedge n}$, the n-fold self-smash of X.

The James-Hopf map $H_k\colon J(X) \to J(X^{\wedge k})$ is defined by

$$H_k(x_1x_2\cdots x_n) = \prod_{1\leq i_1<i_2<\cdots<i_k\leq n} (x_{i_1}\wedge x_{i_2}\wedge \cdots x_{i_k})$$

with right lexicographic order.

Example

$H_2(x_1x_2x_3x_4) = (x_1\wedge x_2)(x_1\wedge x_3)(x_2\wedge x_3)(x_1\wedge x_4)(x_2\wedge x_4)(x_3\wedge x_4).$

James-Hopf Maps

The map H_k is an extension map in the diagram

$$\begin{array}{ccc} J_k(X) & \xrightarrow{\text{pinch}} & J_k(X)/J_{k-1}(X) = X^{\wedge k} \\ \big\downarrow & & \big\downarrow \\ J(X) & \overset{H_k}{\dashrightarrow} & J(X^{\wedge k}) \end{array}$$

- H_k gives a concrete combinatorial construction of Hopf invariants $\Omega\Sigma X \to \Omega\Sigma X^{\wedge k}$.

James-Hopf Maps

- An important application is the EHP fibration $S^n \xhookrightarrow{E} \Omega S^{n+1} \xrightarrow{H_2} \Omega S^{2n+1}$ localized at 2,
- which induces EHP sequence on homotopy groups of spheres.
- **Example.** There is a fibration $S^2 \to \Omega S^3 \to \Omega S^5$ localized at 2. By taking 2-connected cover, there is a fibration $S^3 \to \Omega(S^3\langle 3\rangle) \to \Omega S^5$ and so

$$\cdots \to \pi_{n+2}(S^5) \to \pi_n(S^3) \to \pi_{n+1}(S^3) \to \pi_{n+1}(S^5) \to \cdots$$

for $n \geq 3$.

Computations on homotopy groups: Hiroshi Toda, "Composition Methods in Homotopy. Groups of Spheres," Princeton Univ. Press 1962.

Simplicial sets

- The notion of simplicial set is a generalization of simplicial complex with adding degenerate simplices.
- A simplicial set X refers to a sequence of sets $X = \{X_n\}_{n\geq 0}$, where X_n can be thought as the set of n-simplices, with face operations $d_i\colon X_n \to X_{n-1}$, $0 \leq i \leq n$, and degeneracy operations $s_i\colon X_n \to X_{n+1}$, $0 \leq i \leq n$, satisfying **simplicial identities**.
- **Remark.** Simpicial sets $\longleftrightarrow$ cofunctors from the category of finite ordered sets and order-preserving functions ($f(x) \leq f(y)$ if $x \leq y$) to the category of sets. (Related to, but different from **finite injective objects**.)

Simplicial homotopy theory

- The notion of simplicial set provides a model to work on homotopy theory of *CW*-complexes in a combinatorial way.

- **One of key points by Dan Kan's work:** For a **fibrant** simplicial set (**Kan's complex**) X, the homotopy group of its geometric realization $\pi_n(|X|)$ can be **combinatorially defined** using the data from X, which is the quotient of **spherical** n**-simplicies** ($x \in X_n$ with all $d_i x = *$) modulo **homotopy relations.**

Simplicial groups

- A simplicial group is a simplicial set $G = \{G_n\}_{n\geq 0}$ such that each G_n is a group and faces d_i and degeneracies s_i are group homomorphisms.
- **John Moore's Theorems:** Any simplicial group G is fibrant, and

$$\pi_n(|G|) \cong \frac{\bigcap_{i=0}^{n} \mathrm{Ker}(d_i \colon G_n \to G_{n-1})}{d_0\left(\bigcap_{i=1}^{n+1} (d_i \colon G_{n+1} \to G_n)\right)}.$$

- **Note.** Any (pointed) simplicial map between simplicial groups (not necessary simplicial homomorphism) induces homomorphism on homotopy groups.

The James-Hopf maps on simplicial free monoids

- A point $*$ in a simplicial set X refers to a sequence $\{s_0^n *\}_{n\geq 0}$ with $s_0^n * \in X_n$. (**Note.** A vertex v induces a sequence $([v], [vv], [vvv], \cdots)$)
- Let X be a simplicial set with a basepoint $*$. The **James construction** $J(X)$ is the (simplicial) free monoid generated by X subject to $* = 1$, i.e. $J(X)_n$ is the free monoid generated by X_n subject to $s_0^n * = 1$.
- The geometric realization $|J(X)| \cong J(|X|)$ under compactly generated topology, and the James-Hopf map

$$H_k\colon J(X) \to J(X^{\wedge k})$$

in simplicial setting is defined in the same way as in geometry.

Shortage of simplicial monoids and Milnor's construction

- The simplicial James construction $J(X)$ is a very nice combinatorial model for $\Omega\Sigma|X|$.
- **One shortage** is that $J(X)$ is NOT fibrant. Hence, similar to geometric situation, it is difficult to see the behavior of $H_k\colon J(X) \to J(X^{\wedge k})$ on the homotopy groups.

- **Milnor's construction:** Let X be a simplicial set with a basepoint $*$. Milnor's construction $F[X]$ is the (simplicial) free group generated X subject to $* = 1$. In other words, $F[X]$ is the group completion of $J(X)$.
- The geometric realization $|F[X]| \simeq \Omega\Sigma|X|$.

Fedor Pavutnitskiy's work—Question

- **Question.** How to give a concrete combinatorial construction of the James-Hopf map $H_k\colon F[X] \to F[X^{\wedge k}]$?

- The tricky thing is how to define H_k on the reduced words $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ with some $\epsilon_i = -1$.

Fedor Pavutnitskiy's work—Solution to the Question

- The k-th *combinatorial James-Hopf* map is a natural transformation $H_k : F[-] \to F[(-)^{\wedge k}]$ defined for any pointed (simplicial) set X on reduced words as

$$H_k(x_1^{\epsilon_1} \dots x_n^{\epsilon_n}) = \prod_{(i_1 \dots i_k)} (x_{i_1} \wedge \dots \wedge x_{i_k})^{\epsilon_{i_1} \cdots \epsilon_{i_k}}$$

  here product is taken in lexicographical order with reversing the orders on some parts over sequences of indices $(i_1 \dots i_k)$ such that

$$i_j \leq i_{j+1} - \frac{\epsilon_{i_{j+1}} + 1}{2}$$

  that's it, product is taken over all subsequences $(i_1 \dots i_k)$ of $(1 \dots n)$ with possible repetition of indices, and repetition of index i_j occurs only if corresponding exponent ϵ_{i_j} is negative with under such a case reversing the order of the product of the terms ending with i_j .

Example

Let $w = xyzz^{-1}y^{-1}x^{-1}$. We compute $H_2(w)$. In this case, $x_1 = x, x_2 = y, x_3 = z, x_4^{-1} = z^{-1}, x_5^{-1} = y^{-1}$ and $x_6^{-1} = x^{-1}$. We denote (ij) for $(x_i \wedge x_j)$.

$$
\begin{array}{rll}
 & H_2(x_1x_2x_3x_4^{-1}x_5^{-1}x_6^{-1}) & \\
= & (12)(13)(23) & \\
 & (44)^{+1}(34)^{-1}(24)^{-1}(14)^{-1} & \text{order reversed} \\
 & (55)^{+1}(45)^{+1}(35)^{-1}(25)^{-1}(15)^{-1} & \text{order reversed} \\
 & (66)^{+1}(56)^{+1}(46)^{+1}(36)^{-1}(26)^{-1}(16)^{-1} & \text{order reversed} \\
= & (x \wedge y)(x \wedge z)(y \wedge z) & \\
 & (z \wedge z)^{+1}(z \wedge z)^{-1}(y \wedge z)^{-1}(x \wedge z)^{-1} & \\
 & (y \wedge y)^{+1}(z \wedge y)^{+1}(z \wedge y)^{-1}(y \wedge y)^{-1}(x \wedge y)^{-1} & \\
 & (x \wedge x)^{+1}(y \wedge x)^{+1}(z \wedge x)^{+1}(z \wedge x)^{-1}(y \wedge x)^{-1}(x \wedge x)^{-1} & \\
= & 1 &
\end{array}
$$

Extension of James-Hopf map for James construction

H_k is a natural extension of a combinatorial James-Hopf map for James construction:

$$\begin{array}{ccc} F[X] & \xrightarrow{H_k} & F[X^{\wedge k}] \\ \cup\uparrow & & \cup\uparrow \\ J(X) & \xrightarrow{H_k} & J(X^{\wedge k}) \end{array}$$

Properties of $H_k \colon F[X] \to F[X^{\wedge k}]$

There are many properties of the James-Hopf map $H_k \colon F[X] \to F[X^{\wedge k}]$. One nice property is that H_k nicely fits with (integral or mod p) lower central series:

- **Theorem (Pavutnitskiy-Wu).** Simplicial James-Hopf map $H_m : F[X] \to F[X^{\wedge m}]$ sends lower central series to a weighted one:

$$H_m(\gamma_n) \subset \gamma_n^w,\ H_m(\gamma_n^{[p]}) \subset \gamma_n^{[p],w}$$

Here $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n G, G]$. The **weighted lower central series** of $F[X^{\wedge m}]$ is: $\gamma_n^w(F[X^{\wedge m}]) = F[X^{\wedge m}]$ for $n < m$, and, for $n \geq m$ with $n = qm + s$ and $s < m$,

$$\gamma_n^w(F[X^{\wedge m}]) = \begin{cases} \gamma_q(F[X^{\wedge m}]) & \text{if} \quad s = 0 \\ \gamma_{q+1}(F[X^{\wedge m}]) & \text{if} \quad s > 0 \end{cases}$$

The $\gamma_n^{[p]}$ and $\gamma_n^{[p],w}$ are mod p lower central series and the weighted one.

Unstable Adams spectral sequence

- If X is path-connected, the mod p lower central series of $F[X]$ induces a spectral sequence (unstable Adams spectral sequence or mod p lower central series spectral sequence) convergent to $\pi_*(\Sigma X)$.

- **Note.** $F[X] \simeq \Omega\Sigma X$. For general cases, mod p lower central series of Kan's construction $GY \simeq \Omega Y$ induces a spectral sequence convergent to $\pi_*(Y)$ for simply connected simplicial sets Y.

- We are interested in mod p LCSSS of $F[X]$ for X having two or more cells.

Complexity of homotopy groups of suspended complexes having two or more cells

The simplest case may be the mod p^r Moore space $P^n(p^r) = S^{n-1} \cup_{[p^r]} e^n$. As an illustrative example, the following is a statement on mod 2 Moore space $P^n(2) = S^{n-1} \cup_{[2]} e^n = \Sigma^{n-2}\mathbb{R}\mathrm{P}^2$.

- **Theorem (Ruizhi Huang-Wu).** $P^{n+1}(2)$ is $\mathbb{Z}/2^i$-hyperbolic for each $n \geq 2$ and $i = 1, 2, 3$. Briefly speaking, the function $f(t)$= the number of occurrence of $\mathbb{Z}/2^i$-summands in $\pi_j(P^{n+1}(2))$ with $j \leq t$ has exponential growth.
- **Decompositions** of mod p LCSSS of $F[X]$ may help for reducing the computational complexity in spectral sequence.

The Cohen groups

Briefly speaking the Cohen group $\mathfrak{H}$ is a subgroup of self natural transformations of the functor $\Omega\Sigma$ (on the homotopy categories of path-connected *CW*-complexes) generated by (infinite) product of the following type of maps

$$\Omega\Sigma(X) \xrightarrow{H_k} \Omega\Sigma(X^{\wedge k}) \xrightarrow{\Omega(\alpha)} \Omega\Sigma(X^{\wedge k}) \xrightarrow{\Omega W_k} \Omega\Sigma X, \qquad (1)$$

where $\alpha\colon \Sigma(X^{\wedge k}) \to \Sigma(X^{\wedge k})$ runs over linear combinations of the suspension of the permutations, and W_k is the Whitehead product.

Selick-Wu Program

References.

- Paul Selick and Jie Wu, On natural coalgebra decompositions of tensor algebras and loop suspensions , Memoirs AMS, Vol. 148, No. 701, 2000.
- And generalizations are given in the following up papers.
- **Bott-Samelson Theorem.** Let homology be taken with coefficients in a field. For a simply connected co-H-space Y, $H_*(\Omega Y) \cong T(\Sigma^{-1}\tilde{H}_*(Y))$.
- By using the Cohen group, **Selick-Wu-Theoriault:** Any functorial decomposition of the functor T from $\mathbb{F}_p$-vector spaces to the category of coalgebras induces a functorial decomposition of ΩY for simply connected co-H-spaces Y localized at p with the property that the homology of its factors are given by the corresponding coalgebra factor in the decomposition of T.

The role of $H_k\colon F[X] \to F[X^{\wedge k}]$ in decomposing mod p LCSSS

The maps in Equation (1) can be managed in the following way on Milnor's construction:

$$F[X] \xrightarrow{H_k} F[X^{\wedge k}] \xrightarrow{\tilde{\alpha}} F[X^{\wedge k}] \xrightarrow{\tilde{W}_k} F[X], \tag{2}$$

where $\tilde{\alpha}$ can be chosen as a simplicial homomorphism extending a simplicial map $X^{\wedge k} \to F[X^{\wedge k}]$ and $\tilde{W}_k$ is the simplicial homomorphism extending the iterated commutator mapping $X^{\wedge k} \to F[X]$, $x_1 \wedge x_2 \wedge \cdots \wedge x_k \mapsto [[x_1, x_2], \ldots, x_k]$.

The theorem that H_k preserves weighted lower central series (and weighted mod p lower central series) guarantees that the maps in Equation (2) preserves the lower central series (and mod p lower central series).

Theorem

Here I only state a corollary as a theorem:

- **Theorem (Pavutnitskiy-Wu).** Any natural coalgebra decomposition $T \simeq A \otimes B$ induce a decomposition of spectral sequence

$$E^1_{s,t} = \pi_s(\mathcal{L}^t_{res}(\mathbb{Z}/p[X])) \Longrightarrow \pi_{s+t}(F[X]),$$

$$E^r_{s,t} = E^r_{s,t}(A(\mathbb{Z}/p[X]) \oplus E^r_{s,t}(B(\mathbb{Z}/p[X]))$$

as a functor on sSet_*, with first pages of $E^r_{s,t}(A(\mathbb{Z}/p[X]))$, $E^r_{s,t}(B(\mathbb{Z}/p[X]))$ given by homotopy groups of primitive elements of simplicial coalgebras A and B:

$$E^1_{s,t}(A(\mathbb{Z}/p[X])) = \pi_s(PA(\mathbb{Z}/p[X]))_t,\ E^1_{s,t}(B) = \pi_s(PB(\mathbb{Z}/p[X]))_t,$$

where $\mathbb{Z}/p[X] = \mathbb{Z}/p(X)/\mathbb{Z}/p(*)$

Thank You!