On the homotopy elements $h_0 h_n$

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Let \( p \geq 5 \) be an odd prime. One has the classical Adams spectral sequence (ASS) and the Adams-Novikov spectral sequence (ANSS), they all converge to the stable homotopy groups of spheres.

\[
\{E_r^{s,t}, d_r\} \rightarrow \pi_\ast (S^0_p) \quad E_2 = \text{Ext}_{BP_\ast BP}^{s,t}(BP_\ast, BP_\ast) \\
\Phi \downarrow \Phi
\]

\[
\{E_r^{s,t}, d_r\} \rightarrow \pi_\ast (S^0_p) \quad E_2 = \text{Ext}_A^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)
\]

Between the ANSS and the ASS there is the Thom map \( \Phi \) induced by \( \Phi : BP \rightarrow H\mathbb{Z}/p \).
To detect the $E_2$-terms of the ASS and of the ANSS, one has the following spectral sequences:

\[
\begin{align*}
\text{MSS} & \downarrow \\
H^*(q_n^{-1}Q/(q_0 \cdots q_{n-1})) & \xrightarrow{\text{BSS}} H^*(q_n^{-1}Q/(q_0^\infty \cdots q_{n-1}^\infty)) & \xrightarrow{\text{CSS}} H^*(P, Q) & \xrightarrow{\Phi} Ext_{mA}^{s,t} \\
\text{Alg.} \downarrow \text{NSS} & \downarrow \text{NSS} & \downarrow \text{NSS} & \downarrow \text{NSS} \\
H^*(v_n^{-1}BP_*/(p \cdots v_{n-1})) & \xrightarrow{\text{BSS}} H^*(v_n^{-1}BP_*/(p^\infty \cdots v_{n-1}^\infty)) & \xrightarrow{\text{CSS}} Ext_{BP_*/BP}^{s,t} & \xrightarrow{\Phi} \pi_*(S^0_p) \\
\text{MSS} & \downarrow \text{MSS} & \downarrow \text{MSS} & \downarrow \text{MSS} \\
\end{align*}
\]

where $P = \mathbb{Z}/p[\xi_1, \xi_2, \cdots]$ and $Q = \mathbb{Z}/p[q_0, q_1, \cdots]$. 
The homotopy elements $h_0 h_n$

- One has $\beta_{p^n/p^{n-1}} \in Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$, which is detected by the CSS and $\Phi(\beta_{p^n/p^{n-1}}) = h_0 h_{n+1}$.

$$
\begin{array}{c}
H^*(P, Q) \xrightarrow{\text{CESS}} Ext^2_A \\
\xrightarrow{\text{Alg.}} \xrightarrow{\text{NSS}} \xrightarrow{\Phi} \xrightarrow{\text{ASS}}
\end{array}
$$

$$
H^0(v_2^{-1}BP_*/(p^\infty, v_1^\infty)) \xrightarrow{\text{CSS}} Ext^{2,*}_{BP_*BP} \xrightarrow{\text{ANSS}} \pi_*(S^0)
$$

$$
\beta_{p^n/p^{n-1}} \in Ext^{2,*}_{BP_*BP}(BP_*, BP_*) \subset \pi_*(BP \wedge \widetilde{X}_2)
$$

$$
\Phi
$$

$$
h_0 h_{n+1} \in Ext^{2,*}_A(\mathbb{Z}/p, \mathbb{Z}/p) \subset \pi_*(H \wedge X_2)
$$
• The convergence of $h_0 h_{n+1}$ in the classical ASS (that of $\beta_{p^n/p^n-1}$ in the ANSS) have been being a long standing problem in stable homotopy groups of spheres.

• Let $M$ be the mod $p$ Moore spectrum, $M(1, p^n - 1)$ be the cofiber of $v_1^{p^n-1} : \Sigma^* M \to M$.

Secondary periodic family elements in the ANSS, D. Ravenel

**Theorem** Let $p \geq 5$ be an odd prime. If for some fixed $n \geq 1$,

- the spectrum $M(1, p^n - 1)$ is a ring spectrum,
- $\beta_{p^n/p^n-1}$ is a permanent cycle and
- the corresponding homotopy element has order $p$,

then $\beta_{sp^n/j}$ is a permanent cycle (and the corresponding homotopy element has order $p$) for all $s \geq 1$ and $1 \leq j \leq p^n - 1$.
The homotopy elements \( h_0 h_n \) are in the classical ASS (that of \( \beta_{p^n/p^n-1} \) in the ANSS) have been being a long standing problem in stable homotopy groups of spheres.

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• The convergence of $h_0h_{n+1}$ in the classical ASS (that of $\beta_{p^n/p^n-1}$ in the ANSS) have been being a long standing problem in stable homotopy groups of spheres.

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On the homotopy elements $h_0h_n$
• S. Oka proved that \( M(1, p^n - 1) \) is a ring spectrum.

• From the theorem above and the convergence of \( h_0 h_{n+1} \) one can prove the \( \beta_{p^n/p^{n-1}} \) is a permanent cycle of order \( p \).

\[
\begin{array}{c}
\Sigma^{-1} M \\ \uparrow \beta_{p^n/p^{n-1}} \\
S^0 \\
\downarrow p \\
S^0
\end{array}
\]

People concerned with the triviality of \( v_1^{p^n-1} \beta_{p^n/p^{n-1}} \)

\[
\begin{array}{c}
\Sigma^* M(1, p^n - 1) \\
\downarrow \beta_{p^n/p^{n-1}} \\
\Sigma^{-1} M \\
\downarrow v_1^{p^n-1} \\
\Sigma^* M
\end{array}
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- S. Oka proved that $M(1, p^n - 1)$ is a ring spectrum.
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![Diagram]

People concerned with the triviality of $v_1^{p^n-1}\tilde{\beta}_{p^n/p^{n-1}}$

![Diagram]
- S. Oka proved that $M(1, p^n - 1)$ is a ring spectrum.
- From the theorem above and the convergence of $h_0 h_{n+1}$ one can prove the $\beta_{p^n/p^{n-1}}$ is a permanent cycle of order $p$.

\[
\begin{align*}
\tilde{\beta}_{p^n/p^{n-1}} & \quad S^* \quad 0 \\
\sum^{-1} M & \quad S^0 \quad p \quad S^0
\end{align*}
\]

People concerned with the triviality of $v_1^{p^n-1} \tilde{\beta}_{p^n/p^{n-1}}$

\[
\begin{align*}
\tilde{v}_2^{p^n} & \quad S^* \quad 0 \\
\sum^* M(1, p^n - 1) & \quad \sum^{-1} M \quad \sum^* M
\end{align*}
\]
Toda differential

- $\alpha_1$ and $b_0 = \beta_1$ in $\text{Ext}_{BP_*BP}^*(BP_*, BP_*)$ are permanent cycles in the ANSS, they converges to the homotopy elements $\alpha_1$, $\beta_1$ respectively.

- H. Toda proved that $\alpha_1\beta_1^p = 0$ in $\pi_*(S^0)$.

- The relation $\alpha_1\beta_1^p = 0$ support a Adams differential

$$d_r(x) = \alpha_1 b_0^p.$$ 

It is detected that $x = b_1$, i.e.

$$d_{2p-1}(b_1) = k \cdot \alpha_1 b_0^p$$

- Based on $d_{2p-1}(b_1) = k \cdot \alpha_1 b_0^p$, D. Ravenel proved that

$$d_{2p-1}(b_n) \equiv \alpha_1 b_{n-1}^p$$
The homotopy elements $h_0 h_n$

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- Based on $d_{2p-1}(b_1) = k \cdot \alpha_1 b_0^p$, D. Ravenel proved that

$$d_{2p-1}(b_n) \equiv \alpha_1 b_{n-1}^p$$
• Consider the cofiber sequence

\[ S^0 \xrightarrow{p} S^0 \longrightarrow M \]

which induces a short exact sequence of \( BP \)-homologies

\[ 0 \longrightarrow BP_* \xrightarrow{p} BP_* \longrightarrow BP_* M \longrightarrow 0 \]

• The short exact sequence of \( BP \)-homologies induces a long exact sequence of \( Ext \) groups and it commutes with the Adams differential:

\[ Ext^{s,t}_{BP_* BP}(BP_*, N) \] is denoted by \( Ext^{s,t}(N) \) for short.
\[ \cdots \rightarrow \text{Ext}^1,*(BP_*) \rightarrow \text{Ext}^1,*(BP_*M) \xrightarrow{\delta} \text{Ext}^2,*(BP_*) \rightarrow \cdots \]
\[ \downarrow d_{2p-1} \quad \downarrow d_{2p-1} \quad \downarrow d_{2p-1} \]
\[ \cdots \rightarrow \text{Ext}^{2p},*(BP_*) \rightarrow \text{Ext}^{2p},*(BP_*M) \xrightarrow{\delta} \text{Ext}^{2p+1},*(BP_*) \rightarrow \cdots \]

- There are elements \( v_1 \in \text{Ext}^0,*(BP_*M) \), \( h_{n+1} \in \text{Ext}^1,*(BP_*M) \), \( v_1b_{n-1}^p \in \text{Ext}^{2p},*(BP_*M) \)

\[
\delta(h_{n+1}) = b_n, \quad \delta(v_1b_{n-1}^p) = \alpha_1b_{n-1}^p.
\]
\[
\delta(v_1h_{n+1}) = \beta_{p^n/p^{n-1}}, \quad \delta(v_1^2b_{n-1}^p) = \alpha_2b_{n-1}^p.
\]

- So in the ANSS for the Moore spectrum one has

\[
d_{2p-1}(h_{n+1}) = v_1b_{n-1}^p, \quad d_{2p-1}(v_1h_{n+1}) = v_1^2b_{n-1}^p.
\]

- Applying the connecting homomorphism \( \delta \), one has

\[
d_{2p-1}(\beta_{p^n/p^{n-1}}) = \alpha_2b_{n-1}^p.
\]
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$$\cdots \to Ext^{1,\ast}(BP_\ast) \to Ext^{1,\ast}(BP_\ast M) \xrightarrow{\delta} Ext^{2,\ast}(BP_\ast) \to \cdots$$

$$\downarrow d_{2p-1} \quad \downarrow d_{2p-1} \quad \downarrow d_{2p-1}$$

$$\cdots \to Ext^{2p,\ast}(BP_\ast) \to Ext^{2p,\ast}(BP_\ast M) \xrightarrow{\delta} Ext^{2p+1,\ast}(BP_\ast) \to \cdots$$

- There are elements $v_1 \in Ext^{0,\ast}(BP_\ast M)$, $h_{n+1} \in Ext^{1,\ast}(BP_\ast M)$, $v_1b_{n-1}^p \in Ext^{2p,\ast}(BP_\ast M)$

  $$\delta(h_{n+1}) = b_n,$$

  $$\delta(v_1h_{n+1}) = \beta_{p^n/p^{n-1}},$$

  $$\delta(v_1b_{n-1}^p) = \alpha_1b_{n-1}^p,$$

  $$\delta(v_1^2b_{n-1}^p) = \alpha_2b_{n-1}^p.$$  

- So in the ANSS for the Moore spectrum one has

  $$d_{2p-1}(h_{n+1}) = v_1b_{n-1}^p,$$

  $$d_{2p-1}(v_1h_{n+1}) = v_1^2b_{n-1}^p.$$  

- Applying the connecting homomorphism $\delta$, one has

  $$d_{2p-1}(\beta_{p^n/p^{n-1}}) = \alpha_2b_{n-1}^p.$$  

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\[ \cdots \to Ext^1,*(BP_*) \to Ext^1,*(BP_*M) \xrightarrow{\delta} Ext^2,*(BP_*) \to \cdots \]
\[ \xrightarrow{d_{2p-1}} \]
\[ \cdots \to Ext^{2p},*(BP_*) \to Ext^{2p},*(BP_*M) \xrightarrow{\delta} Ext^{2p+1},*(BP_*) \to \cdots \]

- There are elements \( v_1 \in Ext^0,*(BP_*M) \), \( h_{n+1} \in Ext^1,*(BP_*M) \), \( v_1 b_{n-1}^p \in Ext^{2p},*(BP_*M) \)

\[
\delta(h_{n+1}) = b_n, \\
\delta(v_1 h_{n+1}) = \beta_{p^n/p^{n-1}}, \\
\delta(v_1 b_{n-1}^p) = \alpha_1 b_{n-1}^p, \\
\delta(v_1 b_{n-1}^{2p}) = \alpha_2 b_{n-1}^{2p}.
\]

- So in the ANSS for the Moore spectrum one has

\[
d_{2p-1}(h_{n+1}) = v_1 b_{n-1}^p, \\
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\]

- Applying the connecting homomorphism \( \delta \), one has

\[
d_{2p-1}(\beta_{p^n/p^{n-1}}) = \alpha_2 b_{n-1}^{2p}.
\]
We could NOT prove that

\[ \alpha_2 b_{n-1}^p \in \text{Ext}_{BP_*, BP_*}^{2p+1}(BP_*, BP_*) \]

is non-zero in the \( \text{Ext} \) groups although \( \alpha_1 b_{n-1}^p \) is non-zero.

- \( \alpha_2 b_0^p = 0 \) because \( \alpha_2 \beta_1 = 0 \). And we know that \( \beta_{p/p-1} \) (resp. \( h_0 h_2 \)) survives to \( E_\infty \).

J. Hong and \( \sim \)

Let \( p \geq 5 \) be an odd prime. Then \( \beta_{p^2/p^2-1} \) is a permanent cycle in the ANSS. So \( h_0 h_3 \) is a permanent cycle in the classical ASS.
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Let $p \geq 5$ be an odd prime. Then $\beta_{p^2/p^2-1}$ is a permanent cycle in the ANSS. So $h_0 h_3$ is a permanent cycle in the classical ASS.
The homotopy elements \(h_0 h_n\)

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Further consideration

\[ s\]

\[ g_1 \bullet \xleftarrow{d_{2p-1}} \]

\[ g_3 \bullet \xleftarrow{d_{2p-1}} \]

\[ g_4 \bullet \xleftarrow{d_{2p-1}} \]

\[ g_6 \bullet \xleftarrow{d_{2p-1}} \]

\[ g_7 \bullet g_8 \]

\[ \beta_{p^2/p^2-1} \]

\[ 0 \quad q(p^3 + 1) - 4 \quad q(p^3 + 1) - 3 \quad q(p^3 + 1) - 2 \]

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Small descent SS

- Let $T(m)$ be the Ravenel spectrum characterized by 
  $BP_* T(m) = BP_* [t_1, t_2, \cdots, t_m]$. One has
  \[ S^0 \hookrightarrow T(1) \hookrightarrow T(2) \hookrightarrow \cdots \hookrightarrow T(m) \hookrightarrow \cdots \hookrightarrow BP \]

- Let $X$ be the $(p-1)q$ skeleton of $T(1)$, where $q = 2(p-1)$
  \[ X = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cup \cdots \cup_{\alpha_1} e^{(p-1)q} \]
  and let $\overline{X} = S^0 \cup_{\alpha_1} e^q \cup \cdots \cup_{\alpha_1} e^{(p-2)q}$ be the $(p-2)q$ skeleton of $T(1)$.
  \[ BP_* X = BP_* [t_1]/(t_1^p), \quad BP_* \overline{X} = BP_* [t_1]/(t_1^{p-1}) \]
● One has the cofiber sequences

\[
\begin{align*}
S^0 &\rightarrow X \rightarrow \Sigma^q X \rightarrow S^{(p-1)q} \\
\Sigma^q X &\rightarrow \Sigma^q X \rightarrow S^{pq} \rightarrow \Sigma^{pq} X \\
S^{pq} &\rightarrow \Sigma^{pq} X \rightarrow \Sigma^{(p+1)q} X \rightarrow S^{(2p-1)q}
\end{align*}
\]

● The cofiber sequences gives raise short exact sequences of $BP_*$ homologies

\[
\begin{align*}
0 &\rightarrow BP_* \rightarrow BP_* X \rightarrow BP_* \Sigma^q X \rightarrow 0 \\
0 &\rightarrow BP_* \Sigma^q X \rightarrow BP_* \Sigma^q X \rightarrow BP_* S^{pq} \rightarrow 0 \\
0 &\rightarrow BP_* S^{pq} \rightarrow BP_* \Sigma^{pq} X \rightarrow BP_* \Sigma^{(p+1)q} X \rightarrow 0
\end{align*}
\]

\[\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot
• One has the cofiber sequences

\[
\begin{array}{ccccccc}
S^0 & \longrightarrow & X & \longrightarrow & \Sigma^q \bar{X} & \longrightarrow & S^{(p-1)q} \\
\Sigma^q \bar{X} & \longrightarrow & \Sigma^q X & \longrightarrow & S^{pq} & \longrightarrow & \Sigma^{pq} \bar{X} \\
S^{pq} & \longrightarrow & \Sigma^{pq} X & \longrightarrow & \Sigma^{(p+1)q} \bar{X} & \longrightarrow & S^{(2p-1)q}
\end{array}
\]

\[
\begin{array}{ccccccc}
\vdots & \color{red}{\cdots} & \vdots
\end{array}
\]

• The cofiber sequences gives raise short exact sequences of $BP_*$ homologies

\[
\begin{array}{ccccccc}
0 & \longrightarrow & BP_* & \longrightarrow & BP_* X & \longrightarrow & BP_* \Sigma^q \bar{X} & \longrightarrow & 0 \\
0 & \longrightarrow & BP \Sigma^q \bar{X} & \longrightarrow & BP_* \Sigma^q X & \longrightarrow & BP_* S^{pq} & \longrightarrow & 0 \\
0 & \longrightarrow & BP_* S^{pq} & \longrightarrow & BP_* \Sigma^{pq} X & \longrightarrow & BP_* \Sigma^{(p+1)q} \bar{X} & \longrightarrow & 0 \\
\vdots & \color{red}{\cdots} & \vdots
\end{array}
\]
The homotopy elements $h_0 h_n$
Toda differential
Method of infinite descent
Further consideration

- From the short exact sequences, one gets a long exact sequence

$$0 \to BP_* \to BP_* X \to BP_* \Sigma^q X \to BP_* \Sigma^{pq} X \to BP_* \Sigma^{(p+1)q} X \to \cdots$$

and the long exact sequence induces the small descent spectral sequence.

SDSS, D. Ravenel

Let $X$ be as above. Then there is a spectral sequence converging to $\Ext_{BP_* BP}^{s+u,*}(BP_*, BP_*)$ with $E_1$-term

$$E_1^{s,t,u} = \Ext_{BP_* BP}^{s,t}(BP_*, BP_* X) \otimes E[\alpha_1] \otimes P[\beta_1]$$

where

$$E_1^{s,t,0} = \Ext^{s,t}(BP_* X), \quad \alpha_1 \in E_1^{0,q,1}, \quad \beta_1 \in E_1^{0,pq,2}.$$

and $d_r : E_r^{s,t,u} \to E_r^{s-r+1,t,u+r}$. 
Let \( p \geq 5 \) be an odd prime, then with in \( t - s < q(p^3 + p) \)

\[
E\text{xt}_{BP_*BP}^{s,t}(BP_*, BP_*X \otimes E^2_1) = A \oplus B \oplus C
\]

where \( \otimes E^2_1 \) means except for the first periodic homotopy elements.

Because the total degree \( t - s \) of \( \beta_1 \) is \( pq - 2 = 2p^2 - 2p - 2 \) and that of \( \beta_{p^2/p^2 - 1} \) is \( 4p - 2 \mod pq - 2 \)

\[
\frac{p^2 + 1}{2p^2 - 2p - 2} \quad \frac{\sqrt{2p^4 - 2p^3} + 2p - 4}{2p^4 - 2p^3 - 2p^2} \quad \frac{2p^2 + 2p - 4}{2p^2 - 2p - 2}
\]

\[
\frac{4p^2 - 2}{2p^2 - 2p - 2}
\]
Let $p \geq 5$ be an odd prime, then with $t - s < q(p^3 + p)$

$$
Ext_{BP_*BP}^{s,t}(BP_*, BP_*X \otimes E_1^2) = A \oplus B \oplus C
$$

where $\otimes E_1^2$ means except for the first periodic homotopy elements.

- Because the total degree $t - s$ of $\beta_1$ is $pq - 2 = 2p^2 - 2p - 2$ and that of $\beta_{p^2/p^2 - 1}$ is $4p - 2 \mod pq - 2$

\[
\begin{array}{c}
\frac{p^2}{2p^2 - 2p - 2} + 1 \\
\sqrt{2p^4 - 2p^3} + 2p - 4 \\
2p^4 - 2p^3 - 2p^2 \\
\hline \\
2p^2 + 2p - 4 \\
2p^2 - 2p - 2 \\
\hline
4p - 2
\end{array}
\]
We computed the total degree of the generators in \((A \oplus B \oplus C) \otimes E[\alpha_1]\) mod \(pq - 2\). From which we get the \(E_1\)-term of SDSS
Then we computed the Adams differential and get $d_r(\beta_{p^2/p^2-1}) = 0$. 
Further consideration, where is $\beta_{p/p}$ and $\alpha_2 \beta_{p/p}^p$?

$$H^0(q_2^{-1} Q/(q_0^\infty, q_1^\infty)) \xrightarrow{\text{CSS}} H^*(P, Q) \xrightarrow{\text{CESS}} E_{\infty}^2 \xrightarrow{\alpha} Ext^2_A$$

$$H^0(v_2^{-1} BP_*/(p^\infty, v_1^\infty)) \xrightarrow{\text{CSS}} Ext^2_{BP_* BP} \xrightarrow{\text{NSS}} \pi_*(S^0)$$

2q_1 \xi_1, b_1 \quad 2q_1 \xi_1 \cdot b_1^p \xrightarrow{\text{CESS}} \tilde{\alpha}_2 b_1^p \neq 0

$$\frac{v_1^2}{p}, \frac{v_1^p}{p v_1^p} \xrightarrow{\text{CSS}} \alpha_2, \beta_{p/p}$$

$$\alpha_2 \cdot \beta_{p/p}^p = 0$$

$$d_{2p-1}(\beta_{p^2/p^2-1}) = \alpha_2 \beta_{p/p}^p \text{ and } \beta_{p^2/p^2-1} \text{ survives to } E_\infty \text{ imply } \alpha_2 \beta_{p/p}^p = 0.$$
• $b_1^p = \beta_{p/p}^p \not= 0$ in $Ext^{2p,*}_{BP_*BP}(BP_*, BP_*)$, but $i_*(\beta_{p/p}) = 0$ in $Ext^{2p,*}_{BP_*BP}(BP_*, BP_*X)$

\[ \cdots \longrightarrow Ext^{s-1}(BP_* \Sigma^q X) \xrightarrow{\delta} Ext^s(BP_*) \xrightarrow{i_*} Ext^s(BP_*X) \longrightarrow \cdots \]

• We computed the $E_1$-term $E_1^{s,qp^3,u}$ of the SDSS subject to $s + u = 2p$, which is generated by

\[ \beta_1 h_{11} \gamma_2 b_{20}^{p-3} \quad \beta_1 \alpha_1 b_{20}^{p-3} \eta_p \quad \beta_{p-1}^{p-1} \alpha_1 \eta. \]

This gives a relation $\beta_{p/p} = \beta_1 g$ and

\[ \alpha_2 \beta_{p/p}^p = \alpha_2 \beta_1 g = 0. \]

At prime $p = 5$, $\beta_{5/5}^5 = \beta_1 x_{952}$ and $\alpha_2 \beta_{5/5}^5 = 0$ (D. Ravenel’s Green Book).
The homotopy elements $h_0, h_n$  
Toda differential  
Method of infinite descent  
Further consideration

- $b^p_1 = \beta^p_{p/p} \neq 0$ in $\text{Ext}^{2p, \ast}_{BP \ast BP}(BP \ast, BP \ast)$, but $i_\ast(\beta_{p/p}) = 0$ in $\text{Ext}^{2p, \ast}_{BP \ast BP}(BP \ast, BP \ast X)$

\[
\cdots \to \text{Ext}^{s-1}(BP \ast \Sigma^q X) \xrightarrow{\delta} \text{Ext}^s(BP \ast) \xrightarrow{i_\ast} \text{Ext}^s(BP \ast X) \to \cdots
\]

- We computed the $E_1$-term $E_{1, s, q, p^3, u}$ of the SDSS subject to $s + u = 2p$, which is generated by

\[
\beta_1 h_{11} \gamma_2 b^{p-3}_{20} \quad \beta_1 \alpha_1 b^{p-3}_{20} \eta_p \quad \beta_{p-1}^{\frac{p-1}{2}} \alpha_1 h.
\]

This gives a relation $\beta_{p/p} = \beta_1 g$ and

\[
\alpha_2 \beta_{p/p} = \alpha_2 \beta_1 g = 0.
\]

At prime $p = 5$, $\beta_{5/5}^5 = \beta_1 x_{952}$ and $\alpha_2 \beta_{5/5}^5 = 0$ (D. Ravenel’s Green Book).
Here we guess $\beta_{p/p}^p = \beta_1 h_{11} \gamma_2 b_{20}^{p-3}$ and

$$
\beta_{p/p}^p = \beta_1 h_{11} \gamma_2 b_{20}^{p-3} \\
\beta_{p^2/p^2}^p = \beta_1 h_{21} h_{11} \delta_3 b_{30}^{p-4} \\
\ldots \\
\beta_{p^i/p^i}^p = \beta_1 h_{i,1} h_{i-1,1} \cdots h_{11} \alpha_{i+1}^{(i+2)} b_{i+1.0}^{p-i-2} \\
\ldots \\
\beta_{p^{p-2}/p^{p-2}} = \beta_1 h_{p-2,1} h_{p-3,1} \cdots h_{11} \alpha_{p-1}^{(p)}
$$

where $\alpha_{i+1}^{(i+2)}$ is the $i + 2$-ed Greek letter elements.
Conjecture

- For $i = 0, 1, \cdots, p - 2$

$$\alpha_2 \beta_{p^i/p^i} = \alpha_2 \beta_1 h_{i,1} h_{i-1,1} \cdots h_{11} \alpha_{i+1}^{(i+2)} b_{i+1.0}^{p-i-2} = 0$$

and for $n = 1, 2, \cdots, p - 1$, $\beta_{p^n/p^{n-1}}$ survives to $E_\infty$.

- There is the doomsday for $\beta_{p^n/p^{n-1}}$. If the doomsday for $V(n)$ is 50 years old, $(V(p+1)_{p+1})$ does not exist), the doomsday for $h_0 h_n$ is 100.
Conjecture

- For $i = 0, 1, \cdots, p - 2$

$$\alpha_2 \beta_{p^i/p^i} = \alpha_2 \beta_1 h_{i,1} h_{i-1,1} \cdots h_{11} \alpha_{i+1}^{(i+2)} b_{p-i-2}^{p-i-2} = 0$$

and for $n = 1, 2, \cdots, p - 1$, $\beta_{p^n/p^{n-1}}$ survives to $E_\infty$.

- There is the doomsday for $\beta_{p^n/p^{n-1}}$. If the doomsday for $V(n)$ is 50 years old, $(V(p+1/2)$ does not exist), the doomsday for $h_0 h_n$ is 100.
Conjecture

- For $i = 0, 1, \ldots, p - 2$
  
  \[
  \alpha_2 \beta_{p^i/p^i} = \alpha_2 \beta_1 h_{i,1} h_{i-1,1} \cdots h_{1,1} \alpha_{i+1}^{(i+2)} b_{i+1.0}^{p-i-2} = 0
  \]
  
  and for $n = 1, 2, \cdots, p - 1$, $\beta_{p^n/p^n-1}$ survives to $E_\infty$.

- There is the doomsday for $\beta_{p^n/p^n-1}$. If the doomsday for $V(n)$ is 50 years old, $(V(p+1/2) \text{ does not exist})$, the doomsday for $h_0h_n$ is 100.
Conjecture

For $n \geq p - 1$, $\alpha_2 \beta_{p^n/p^n} \neq 0$ and

$$d_{2p-1}(\beta_{p^{n+1}/p^{n+1}} - 1) = \alpha_2 \beta_{p^n/p^n}^p.$$ 

From $\beta_{p^p/p^{p-1}}$, $\beta_{p^n/p^n-1}$ does not exist and from $h_0 h_{p+1}$, $h_0 h_n$ does not exist.
Thank you!