HHR KERVAIRE INVARIANT PROBLEM SET

Abstract. This note contains some exercises and open problems related to HHR's solution to the Kervaire invariant problem. The exercises are meant to complement the lectures. In particular, in order to work on some of the open problems, it is necessary to be comfortable with some slice spectral sequence computations.

1. Exercises

1.1. Exercise 1: SliceSS($MU_{\mathbb{R}}$) and SliceSS($BP_{\mathbb{R}}$).

Recall that

$$\pi_* MU = \mathbb{Z}[x_1, x_2, x_3, \ldots], \ \ |x_i| = 2i.$$

If we work 2-locally, MU splits as a wedge of suspensions of BP (the Brown–Peterson spectrum), whose homotopy groups are

$$\pi_*(BP) = \mathbb{Z}_{(2)}[v_1, v_2, v_3, \ldots],$$

where $v_i := x_{2^i-1}$. Computationally, the spectrum BP is often easier to work with than MU because its homotopy groups are smaller. Moreover, no information is lost by going from MU to BP.

The same story is true C_2-equivariantly. Recall from Danny's talk that

$$\pi^{C_2}_{*\rho_{C_2}} MU_{\mathbb{R}} = \mathbb{Z}_{(2)}[\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots].$$

It turns out that if we look 2-locally, the spectrum $MU_{\mathbb{R}}$ splits as a wedge of suspensions (by regular representation spheres) of $BP_{\mathbb{R}}$. We have

$$\pi^{C_2}_{*\rho_{C_2}} BP_{\mathbb{R}} = \mathbb{Z}_{(2)}[\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots].$$

The following exercises are supposed to guide you through some of the C_2-slice spectral sequence computations. If you are stuck, refer to Mingcong's talk notes.

1. Write down the slices of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$. If you are stuck, refer to Danny's talk notes and Section 6 of HHR (the Slice Theorem).
2. By writing down the C_2-equivariant cell decomposition of $S^{n\rho_{C_2}}$, compute the homotopy groups
$$\pi_*(H\underline{\mathbb{Z}} \wedge S^{n\rho_{C_2}}) = H^{C_2}_*(S^{n\rho_{C_2}}; \underline{\mathbb{Z}})$$
by writing down the equivariant cell complex of $S^{n\rho_{C_2}}$.
3. From (1) and (2), deduce the E_1-page of the slice spectral sequence for $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$.
4. Show that if X is a C_2-spectrum, then
$$\Phi^{C_2}(X) \simeq (\widetilde{E}C_2 \wedge X)^{C_2} \simeq (a_\sigma^{-1} X)^{C_2}.$$
From the construction of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$, we have the following equivalences:
$$\begin{aligned} \Phi^{C_2}(MU_{\mathbb{R}}) &= MO \\ \Phi^{C_2}(BP_{\mathbb{R}}) &= H\mathbb{F}_2. \end{aligned}$$

(5) By inverting a_σ^{-1} in the slice spectral sequence and (4), deduce all the differentials in SliceSS($MU_{\mathbb{R}}$) and SliceSS($BP_{\mathbb{R}}$).

1.2. **Exercise 2: SliceSS($k_{\mathbb{R}}$) and SliceSS($K_{\mathbb{R}}$).** In this exercise, we are going to compute the C_2-SliceSS of Atiyah's Real K-theory $K_{\mathbb{R}}$ and its connective version $k_{\mathbb{R}}$.

Recall from Mingcong's talk that

$$k_{\mathbb{R}} = BP_{\mathbb{R}}/(\bar{v}_2, \bar{v}_3, \ldots),$$

and

$$K_{\mathbb{R}} = \bar{v}_1^{-1} BP_{\mathbb{R}}/(\bar{v}_2, \bar{v}_3, \ldots).$$

(1) Write down the slices of $k_{\mathbb{R}}$ and $K_{\mathbb{R}}$. (Note: $K_{\mathbb{R}}$ have negative slices.)
(2) Compute the E_1-page of the slice spectral sequence for $k_{\mathbb{R}}$ and $K_{\mathbb{R}}$.
(3) The quotient map

$$MU_{\mathbb{R}} \longrightarrow k_{\mathbb{R}},$$

induces a map

$$\text{SliceSS}(MU_{\mathbb{R}}) \longrightarrow k_{\mathbb{R}}$$

of spectral sequences. Use the fact that we have deduced all the differentials in SliceSS($MU_{\mathbb{R}}$) in Exercise 1 to deduce the differentials in SliceSS($k_{\mathbb{R}}$).
(4) The map

$$k_{\mathbb{R}} \longrightarrow K_{\mathbb{R}}$$

induces a map

$$\text{SliceSS}(k_{\mathbb{R}}) \longrightarrow \text{SliceSS}(K_{\mathbb{R}})$$

of spectral sequences. Use the answer in (3) to deduce all the differentials in SliceSS($K_{\mathbb{R}}$). Observe that the final result is 8 periodic.
(5) Since $K_{\mathbb{R}}^{C_2} = KO$, we have used the slice spectral sequence to compute the homotopy groups of KO, the real orthogonal K-theory. Show that $K_{\mathbb{R}}^{C_2} \simeq K_{\mathbb{R}}^{hC_2}$.
(6) Prove that for the connective Real K-theory,

$$k_{\mathbb{R}}^{C_2} \not\simeq k_{\mathbb{R}}^{hC_2}.$$

In other words, the homotopy fixed point theorem is not true for connective Real K-theory. (Hint: compare the slice spectral sequence with the homotopy fixed point spectral sequence)

1.3. **Exercise 3: Real Johnson–Wilson Theory $E_{\mathbb{R}}(n)$.** There is a more general construction that you can make that are the higher height analogues of Atiyah's Real K-theory, starting from $BP_{\mathbb{R}}$:

$$\begin{aligned} BP_{\mathbb{R}}\langle n\rangle &:= BP_{\mathbb{R}}/(\bar{v}_{n+1}, \bar{v}_{n+2}, \ldots) \\ E_{\mathbb{R}}(n) &:= \bar{v}_n^{-1} BP_{\mathbb{R}}\langle n\rangle. \end{aligned}$$

The spectrum $E_{\mathbb{R}}(n)$ is called Real–Johnson–Wilson theory. It is the C_2-equivariant spectrum lifting $E(n)$, the classical Johnson–Wilson theory. See a list of papers by Kitchloo, Wilson, and Lorman.

(1) Use the induced map
$$\text{SliceSS}(BP_{\mathbb{R}}) \longrightarrow \text{SliceSS}(BP_{\mathbb{R}}\langle n\rangle)$$
and
$$\text{SliceSS}(BP_{\mathbb{R}}) \longrightarrow \text{SliceSS}(E_{\mathbb{R}}(n))$$
to compute the slice spectral sequence of $BP_{\mathbb{R}}\langle n\rangle$ and $E_{\mathbb{R}}(n)$.
(2) Show that $E_{\mathbb{R}}(n)^{C_2} \simeq E_{\mathbb{R}}(n)^{hC_2}$, but $BP_{\mathbb{R}}\langle n\rangle^{C_2} \not\simeq BP_{\mathbb{R}}\langle n\rangle^{hC_2}$.
(3) Prove that
$$MU_{\mathbb{R}}^{C_2} \simeq MU_{\mathbb{R}}^{hC_2}$$
and
$$BP_{\mathbb{R}}^{C_2} \simeq BP_{\mathbb{R}}^{hC_2}$$
by analyzing the map of spectral sequences
$$\text{SliceSS}(MU_{\mathbb{R}}) \longrightarrow \text{HFPSS}(MU_{\mathbb{R}})$$
$$\text{SliceSS}(BP_{\mathbb{R}}) \longrightarrow \text{HFPSS}(BP_{\mathbb{R}}).$$
Here, HFPSS is the C_2-homotopy fixed point spectral sequence. The moral of the story here is that for n finite, none of $BP_{\mathbb{R}}\langle n\rangle$ are cofree. But when $n \to \infty$, $BP_{\mathbb{R}}$ becomes cofree.

The spectrum $E_{\mathbb{R}}(n)$ have been studied extensively by Hu–Kriz, Kitchloo–Wilson, and Kitchloo–Lorman–Wilson. For instance, Kitchloo and Wilson computed $E_{\mathbb{R}}(n)^*_{C_2}(\mathbb{R}P^n)$ and used this computation to prove some non-immersion results of real projective spaces. There are a variety of open problems surrounding $E_{\mathbb{R}}(n)$. We will describe them below in the open problems section.

Here are some useful references regarding $E_{\mathbb{R}}(n)$:

(1) Kitchloo–Wilson: "On fibrations related to real spectra".
(2) Kitchloo–Wilson: "The second real Johnson–Wilson theory and non-immersions of $\mathbb{R}P^n$ part I, part II".
(3) Kitchloo–Lorman–Wilson: "Landweber flat real pairs and $E_{\mathbb{R}}(n)$-cohomology"

2. Open Problems (to be expanded and refined)

2.1. Equivariant commutativity of $E_{\mathbb{R}}(n)$. The underlying spectrum of $E_{\mathbb{R}}(n)$ is $E(n)$, the classical Johnson–Wilson theory. We know that $E(n)$ is a homotopy commutative ring spectrum.

The goal of this project is to investigate whether $E_{\mathbb{R}}(n)^{C_2}$ is a homotopy commutative ring spectrum. If it is, then this would imply that $E_{\mathbb{R}}(n)$ is a C_2-equivariant homotopy commutative ring spectrum.

Kitchloo–Wilson, in their paper "multiplicative structure on Real Johnson–Wilson theory" have shown that $E_{\mathbb{R}}(n)$ is a homotopy commutative ring spectrum up to phantom maps. Can we do better and show they are actually homotopy commutative?

2.2. Homology of a point. As we have seen in HHR's proof, the Gap theorem relies on a crucial vanishing result in $\pi_{\bigstar}^{C_{2^n}} H\underline{\mathbb{Z}}$.

Compute $\pi_{\bigstar}^{G} H\underline{\mathbb{Z}}$ and $\pi_{\bigstar}^{G} H\underline{\mathbb{A}}$ for different groups (and different Eilenberg Maclane objects). Do similar vanishing results hold?

2.3. **Slices of suspensions of Eilenberg Maclane spectra.** In Danny's talk, we have seen that the slice tower of X and the slice tower of $\Sigma^{k\rho_G} X$ are intimately related. In general, knowing the slice tower of X does not give the slice tower of $\Sigma^V X$.

Question 2.1. What is the relationship between the slice tower of X and the slice tower of suspensions of X?

In particular, when X is the Eilenberg Maclane spectrum, what is the relationship between the slice tower of $H\underline{\mathbb{Z}}$ and the slice tower of $\Sigma^V H\underline{\mathbb{Z}}$?

See Hill–Hopkins–Ravenel: "The slice spectral sequence for certain $RO(C_{p^n})$-graded suspensions of $H\underline{\mathbb{Z}}$" for some work that has been done in this direction.

2.4. **Computing $E_{\mathbb{R}}(n)^*_{C_2}(\mathbb{R}P^m)$ by using the slice spectral sequence.** Kitchloo–Wilson computed $\pi_*^{C_2} E_{\mathbb{R}}(n)$ and $E_{\mathbb{R}}(n)^*_{C_2}(\mathbb{R}P^m)$ by a Bockstein spectral sequence.

Compute $E_{\mathbb{R}}(n)^*_{C_2}(\mathbb{R}P^m)$ by using the slice spectral sequence? A good place to start is to recover Kitchloo–Wilson's computation of $E_{\mathbb{R}}(2)^*_{C_2}(\mathbb{R}P^m)$ by using the slice spectral sequence.

Once these cohomology theories are computed, is it possible to obtain better non-immersion results of real projective spaces?

2.5. **Slice tower of S^0.** In the motivic slice story, the slices of S^0 are known (see Marc Levine's paper "The Adams–Novikov spectral sequence and Voevodsky's slice tower"). For any group G (even when $G = C_2$), the slice tower of S^0 is unknown. What is the slice tower of S^0? How is the slice spectral sequence related to the classical spectral sequences (AdamsSS, Adams–Novikov SS) that computes the stable homotopy groups of spheres?

2.6. **Hurewicz images of $MU^{((G))}$?** From Hill–Hopkins–Ravenel's solution, we know that the Kervaire invariant elements are detected by the G-fixed points of $MU^{((G))}$. Are there any other elements in the stable homotopy groups of spheres that are detected by $\pi_*^G MU^{((G))}$?
References:

(1) Hill: "on the fate of η^3 in higher analogues of Real bordism"
(2) Li–Shi–Wang–Xu: "Hurewicz images of Real bordism theory and Real Johnson–Wilson theories"

2.7. **Equivariant Mahowald Invariant.** Given two irreducible representations V and W of G, there are associated Euler classes $a_V \in \pi^G_{-V} S^0$ and $a_W \in \pi^G_{-W} S^0$. Given a power of a_V, how divisible is it by powers of a_W? In other words, for what $p, q \geq 2$ do the following diagram exist:

$$\begin{tikzcd}
S^{pV} \arrow[dr, "\gamma"] & \\
S^0 \arrow[u, "a_V^p"] \arrow[r, "a_W^q"'] & S^{qW}
\end{tikzcd}$$

This has connections to the classical Mahowald invariant and in general, for certain groups G, this has applications in geometric topology.
References:

(1) Crabb: "periodicity in $\mathbb{Z}/4$-equivariant stable homotopy theory"
(2) Schmidt: "Spin 4-manifolds and Pin(2)-equivariant homotopy theory"

(3) Stolz: "the level of real projective spaces"
(4) Hopkins–Lin–Shi–Xu: "intersection forms of spin 4-manifolds and the Pin(2)-equivariant Mahowald invariant"

2.8. **Computing the equivariant dual Steenrod algebra.** The Hill–Hopkins–Ravenel reduction theorem states that

$$MU^{((G))}/(G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots) \simeq H\underline{\mathbb{Z}}.$$

How can we leverage this fact to compute the equivariant Steenrod algebra

$$H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}?$$