Equivariant spectra and Mackey functors

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Suppose G is a finite group, and let Orb_G be the orbit category of G. By Elmendorf's theorem (cf. Talk 2.2), there is an equivalence between the homotopy theory of G-spaces, and the homotopy theory of topological presheaves over Orb_G. In this note, we introduce Mackey functors and explain Guillou and May's version of Elmendorf's theorem for G-spectra. This result gives an algebraic perspective on equivariant spectra.[1]

1 Mackey functors

Spectra are higher algebraic analogues of abelian groups. Thus, we begin by considering the fixed points of G-modules, and then we turn to G-spectra in general.

Suppose $(M, +, 0)$ is a G-module. By neglect of structure, M is a G-set, and therefore its fixed points $M^H \cong \mathbf{Set}^G(G/H, M)$ form a presheaf over Orb_G. Spelled out, we have an inclusion $M^K \hookleftarrow M^H$ for every inclusion $K \subset H \subset G$ of subgroups, and an isomorphism $g(-) : M^H \to M^{gHg^{-1}}$ for every element $g \in G$ and subgroup $H \subset G$.

The extra additive structure on M induces extra additive structure on the fixed points of M. Suppose $H \subset G$ is a subgroup. By adjunction, an H-fixed point $G/H \to M$ is equivalent to a G-linear map $\mathbb{Z}[G/H] \to M$, and thus $M^H \cong \hom_{\mathbb{Z}[G]}(\mathbb{Z}[G/H], M)$ is an abelian group. More interestingly, for any inclusion of subgroups $K \subset H \subset G$, there is a G-linear map $\mathbb{Z}[G/H] \to \mathbb{Z}[G/K]$ that represents the element $\sum_{H/K} rK \in \mathbb{Z}[G/K]^H$. Pulling back along it yields an additive "transfer map"

$$\mathrm{tr}_K^H(x) = \sum_{rK \in H/K} rx : M^K \to M^H.$$

This constellation of data is an example of a Mackey functor. We give an axiomatic definition now, and we reformulate it in conceptual terms later.

Definition 1.1. A *G-Mackey functor* $\underline{M}$ consists of

1. an abelian group $\underline{M}(H)$ for every subgroup $H \subset G$, and

[1]Many thanks to Mike Hill for his suggestions and comments on this material.

2. additive restriction, transfer, and conjugation homomorphisms

$$\mathrm{res}^H_K : \underline{M}(H) \to \underline{M}(K) \quad \mathrm{tr}^H_K : \underline{M}(K) \to \underline{M}(H) \quad c_g : \underline{M}(H) \to \underline{M}(gHg^{-1})$$

for all subgroups $K \subset H \subset G$ and elements $g \in G$,

such that:

(i) $\mathrm{res}^H_H = \mathrm{tr}^H_H = \mathrm{id}$ for all subgroups $H \subset G$,

(ii) $c_h : \underline{M}(H) \to \underline{M}(H)$ is the identity map for all subgroups $H \subset G$ and $h \in H$,

(iii) $\mathrm{res}^L_K \mathrm{res}^H_L = \mathrm{res}^H_K$ and $\mathrm{tr}^H_L \mathrm{tr}^L_K = \mathrm{tr}^H_K$ for all subgroups $K \subset L \subset H \subset G$,

(iv) $c_g c_h = c_{gh}$ for all elements $g, h \in G$,

(v) $c_g \mathrm{res}^H_K = \mathrm{res}^{gHg^{-1}}_{gKg^{-1}} c_g$ and $c_g \mathrm{tr}^H_K = \mathrm{tr}^{gHg^{-1}}_{gKg^{-1}} c_g$ for all subgroups $K \subset H \subset G$ and elements $g \in G$, and

(vi) the *double coset formula*

$$\mathrm{res}^H_L \mathrm{tr}^H_K = \sum_{a \in L \backslash H / K} \mathrm{tr}^L_{L \cap aKa^{-1}} c_a \mathrm{res}^K_{a^{-1}La \cap K}$$

holds for all subgroups $K, L \subset H \subset G$.

The double coset formula is an algebraic incarnation of the orbit decomposition $\mathrm{res}^H_L H/K \cong \coprod_{a \in L \backslash H/K} L/(L \cap aKa^{-1})$.

A morphism $f : \underline{M} \to \underline{N}$ of Mackey functors is a family of group homomorphisms $f(H) : \underline{M}(H) \to \underline{N}(H)$ that preserve all restriction, transfer, and conjugation maps. We write $\mathbf{Mack}(G)$ for the category of all G-Mackey functors.

When G is trivial, a G-Mackey functor is the same thing as an abelian group, so it makes sense to ask how ordinary algebraic notions equivariantize. To start, let's consider the integers $\mathbb{Z}$. This is the free abelian group on a single generator. We shall see that the free G-Mackey functor on a single generator (in the G-component) is the *Burnside Mackey functor* $\underline{A}$, described below. Recall that the *Grothendieck group* or *group completion* of a commutative monoid M is the initial abelian group receiving an additive map from M.

Example 1.2. For any subgroup $H \subset G$, let $\underline{A}(H)$ be the Grothendieck group of the commutative monoid of isoclasses of finite H-sets, under disjoint union. Then $\underline{A}$ is a Mackey functor. Its restrictions are induced by $\mathrm{res}^H_K : \mathbf{Set}^H \to \mathbf{Set}^K$, its transfers are induced by $\mathrm{ind}^H_K = H \times_K (-) : \mathbf{Set}^K \to \mathbf{Set}^H$, and its conjugations are induced by $g(-) : \mathbf{Set}^H \to \mathbf{Set}^{gHg^{-1}}$. When G is trivial, $\underline{A} \cong \mathbb{Z}$.

It will be convenient to repackage the data in Definition 1.1 as a certain kind of functor. We start by describing a domain category that encodes axioms (i) – (vi).

Definition 1.3. The *Lindner category* $\mathscr{B}_G^+$ is defined as follows. An object of $\mathscr{B}_G^+$ is a finite G-set. A morphism from X to Y in $\mathscr{B}_G^+$ is a span diagram $X \leftarrow U \rightarrow Y$ of finite G-sets, modulo the equivalence relation that identifies $X \leftarrow U \rightarrow Y$ with $X \leftarrow U' \rightarrow Y$ if there is an isomorphism $U \cong U'$ that makes both triangles commute. The composite of $X \leftarrow U \rightarrow Y$ and $Y \leftarrow V \rightarrow Z$ is represented by the span $X \leftarrow W \rightarrow Z$ constructed by pulling back and composing.

$$\begin{tikzcd}
& & W \arrow[dl] \arrow[dr] \arrow[ddll, bend right] \arrow[ddrr, bend left] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& U \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

The hom sets of $\mathscr{B}_G^+$ are commutative monoids. The sum of $X \leftarrow U \rightarrow Y$ and $X \leftarrow V \rightarrow Y$ is represented by $X \leftarrow U \sqcup V \rightarrow Y$, and the class of $X \leftarrow \varnothing \rightarrow Y$ is the additive identity. The *Burnside category* $\mathscr{B}_G$ is obtained from the Lindner category $\mathscr{B}_G^+$ by applying group completion homwise.

Remark 1.4. The category $\mathscr{B}_G^+$ is also sometimes called the Burnside category, but we shall reserve this name exclusively for $\mathscr{B}_G$.

Note that $\mathscr{B}_G$ is an additive category, meaning it is enriched over the category of abelian groups, and has all finitary biproducts. The empty set $\varnothing$ is the zero object, and the disjoint union $X \sqcup Y$ is the product and coproduct of X and Y in $\mathscr{B}_G$. Note also that the category $\mathscr{B}_G$ is self-dual.

We now give a second definition of a Mackey functor. Let **Ab** denote the category of abelian groups.

Definition 1.5. A *G-Mackey functor* $\underline{M}$ is a contravariant **Ab**-enriched functor from the Burnside category $\mathscr{B}_G$ to the category **Ab** of abelian groups.

Remark 1.6. We could just as well define Mackey functors to be covariant **Ab**-enriched functors, because $\mathscr{B}_G$ is self-dual. Note that a Mackey functor automatically sends disjoint unions $X \sqcup Y$ of finite G-sets to direct sums because it respects composition and the additive enrichment.

Definitions 1.1 and 1.5 are equivalent. If $\underline{M} : \mathscr{B}_G^{\mathrm{op}} \rightarrow \mathbf{Ab}$ is a Mackey functor in the second sense, then we recover a Mackey functor in the first sense by setting

$$\begin{aligned}
\underline{M}(H) &= \underline{M}(G/H) \\
\mathrm{res}_K^H &= \underline{M}(G/K \xleftarrow{\mathrm{id}} G/K \xrightarrow{\pi} G/H) \\
\mathrm{tr}_K^H &= \underline{M}(G/H \xleftarrow{\pi} G/K \xrightarrow{\mathrm{id}} G/K) \\
c_g &= \underline{M}(G/gHg^{-1} \xleftarrow{\mathrm{id}} G/gHg^{-1} \xrightarrow{(-)g} G/H) \\
&= \underline{M}(G/gHg^{-1} \xleftarrow{(-)g^{-1}} G/H \xrightarrow{\mathrm{id}} G/H).
\end{aligned}$$

The double coset formula arises from the pullback squares that define composition in $\mathscr{B}_G^+$, and the values above determine $\underline{M} : \mathscr{B}_G^{\mathrm{op}} \rightarrow \mathbf{Ab}$ by functoriality and additivity.

Example 1.7. The representable Mackey functor $\mathscr{B}_G(-, G/G) : \mathscr{B}_G^{\mathrm{op}} \to \mathbf{Ab}$ is isomorphic to $\underline{A}$, because there is an equivalence $\mathbf{Set}^G/(G/H) \simeq \mathbf{Set}^H$ that sends a G-map $p : X \to G/H$ to its fiber over eH. From here, the Yoneda lemma implies that $\underline{A}$ is free on a generator in G/G. In general, the representable Mackey functor $\mathscr{B}_G(-, \coprod_{i \in I} G/H_i)$ is free on the set of generators $\{x_i \mid i \in I\}$, where x_i is regarded as an element in the G/H_i component.

The category $\mathbf{Mack}(G)$ is an abelian category. Kernels and cokernels are computed pointwise, and similarly for more general small limits and colimits. There is also a tensor product of Mackey functors, given by Day convolution.

Definition 1.8. The *box product* $\underline{M} \boxtimes \underline{N}$ of two Mackey functors $\underline{M}, \underline{N} : \mathscr{B}_G^{\mathrm{op}} \rightrightarrows \mathbf{Ab}$ is the left Kan extension of $\otimes \circ (\underline{M} \times \underline{N}) : \mathscr{B}_G^{\mathrm{op}} \times \mathscr{B}_G^{\mathrm{op}} \to \mathbf{Ab}$ along the cartesian product functor $\times$.

$$\begin{tikzcd} \mathscr{B}_G^{\mathrm{op}} \times \mathscr{B}_G^{\mathrm{op}} \arrow[r, "\underline{M} \times \underline{N}"] \arrow[d, "\times"'] & \mathbf{Ab} \times \mathbf{Ab} \arrow[r, "\otimes"] & \mathbf{Ab} \\ \mathscr{B}_G^{\mathrm{op}} \arrow[urr, "\underline{M} \boxtimes \underline{N} = \mathrm{Lan}_\times (\otimes \circ \underline{M} \times \underline{N})"'] & & \end{tikzcd}$$

By definition, the box product has the following universal property

$$\hom(\underline{M} \boxtimes \underline{N}, \underline{P}) \cong \hom(\otimes \circ (\underline{M} \times \underline{N}), \underline{P} \circ \times),$$

and for any finite G-set $X \in \mathscr{B}_G$, the coend formula implies

$$(\underline{M} \boxtimes \underline{N})(X) \cong \int^{(Y,Z) \in \mathscr{B}_G \times \mathscr{B}_G} \underline{M}(Y) \otimes \underline{N}(Z) \otimes \mathscr{B}_G(X, Y \times Z).$$

The box product makes $\mathbf{Mack}(G)$ into a closed symmetric monoidal category, in which the Burnside Mackey functor $\underline{A}$ is the unit. The Mackey functor-valued hom $\underline{\hom}(\underline{M}, \underline{N})$ is defined by an end dual to the coend for $\underline{M} \boxtimes \underline{N}$.

A (commutative) monoid with respect to the box product is called a (commutative) *Green functor*. A Green functor $\underline{R}$ behaves like a ring to some extent, but there is an asymmetry between its addition and multiplication. Every component $\underline{R}(H)$ of a Green functor is a ring, but we are not guaranteed any multiplicative transfer (norm) maps $n_K^H : \underline{R}(K) \to \underline{R}(H)$. Roughly speaking, a *Tambara functor* is a commutative Green functor equipped with multiplicative norms n_K^H for all subgroups $K \subset H \subset G$, which satisfy the double coset formula and a distributive law over the transfers. Blumberg and Hill's *incomplete Tambara functors* [2] interpolate between these two extremes.

For further introductions to Mackey functors, we recommend [11][2] and [12].

[2]This is also available online as arXiv:1405.1770.

2 Equivariant stable homotopy groups

Mackey functors play the role of abelian groups in equivariant homotopy theory. Just as the stable homotopy groups of spectra are abelian groups, the stable homotopy groups of genuine G-spectra are Mackey functors. Moreover, Brun [3, §7.2] has shown that π_0 of a genuine commutative ring G-spectrum is a Tambara functor, and Blumberg and Hill generalize this result to less structured ring spectra [2, Theorem 1.6].

That being said, a wrinkle appears in the construction of transfer maps in equivariant homotopy theory. To see the problem, suppose that M is a G-module, and consider the transfer $\mathrm{tr}_e^G(x) = \sum_{r\in G} rx : M \to M^G$ once more. The element $\mathrm{tr}_e^G(x)$ is G-fixed because for any $g \in G$, we have

$$g\sum_{r\in G} rx = \sum_{r\in G} grx = \sum_{r\in G} rx$$

by the strict G-equivariance and commutativity of addition in M. In equivariant homotopy theory, is it standard to assume that G acts strictly, but it is completely unreasonable to assume that addition is strictly commutative. Nonequivariantly, every connected, strictly commutative topological monoid splits as product of Eilenberg-MacLane spaces.

Fortunately, we do not need a strictly G-equivariant and commutative sum to construct transfers. Suppose $x \in M$ is an element, and regard it as an additive map $x : \mathrm{res}_e^G \mathbb{Z}[G/G] \to \mathrm{res}_e^G M$. Then there is an adjoint map of G-modules $x : \mathbb{Z}[G/G] \to \mathrm{coind}_e^G \mathrm{res}_e^G M$, and there is an isomorphism $\mathrm{coind}_e^G \cong \mathrm{ind}_e^G$. Composing with the counit $\varepsilon : \mathrm{ind}_e^G \mathrm{res}_e^G M \to M$ yields a G-map

$$\mathbb{Z}[G/G] \to \mathrm{coind}_e^G \mathrm{res}_e^G M \cong \mathrm{ind}_e^G \mathrm{res}_e^G M \xrightarrow{\varepsilon} M$$

that represents $\mathrm{tr}_e^G(x) \in M^G$. We think of the composite

$$\Sigma_G : \mathrm{coind}_e^G \mathrm{res}_e^G M \cong \mathrm{ind}_e^G \mathrm{res}_e^G M \to M$$

as a G-fold twisted sum. Analogous maps naturally appear in homotopy commutative settings.

Suppose $H \subset G$ is a subgroup. The restriction functor $\mathrm{res}_H^G : \mathbf{Sp}^G \to \mathbf{Sp}^H$ has left and right adjoints given by induction ind_H^G and coinduction coind_H^G, and the canonical G-map

$$\mathrm{ind}_H^G X \to \mathrm{coind}_H^G X$$

is the *Wirthmüller isomorphism*. It is a equivalence of genuine G-spectra. If X is a G-spectrum, then combining the counit $\varepsilon : X \wedge G/H_+ \cong \mathrm{ind}_H^G \mathrm{res}_H^G X \to X$ with the stable equivalence $\mathrm{ind}_H^G \mathrm{res}_H^G X \simeq \mathrm{coind}_H^G \mathrm{res}_H^G X$ produces a G/H-fold twisted sum

$$\Sigma_{G/H} : \mathrm{coind}_H^G \mathrm{res}_H^G X \simeq \mathrm{ind}_H^G \mathrm{res}_H^G X \to X.$$

These maps give rise to the transfers in the homotopy groups of X.

Definition 2.1. Suppose X is a genuine G-spectrum and $n \in \mathbb{Z}$. The nth homotopy groups of X form a Mackey functor whose H-component is

$$\pi_n^H X = [S^n \wedge G/H_+, X]^G.$$

By duality, there is an isomorphism $[S^n, X \wedge G/H_+]^G \cong [S^n \wedge G/H_+, X]^G$, and the restrictions, transfers, and conjugations for $\underline{\pi}_n X$ are defined by

$$\begin{aligned}
\mathrm{res}_K^H &= (\mathrm{id} \wedge \pi_+)^* : [S^n \wedge G/H_+, X]^G \to [S^n \wedge G/K_+, X]^G \\
\mathrm{tr}_K^H &= (\mathrm{id} \wedge \pi_+)_* : [S^n, X \wedge G/K_+]^G \to [S^n, X \wedge G/H_+]^G \\
c_g &= (\mathrm{id} \wedge (-)g_+)^* : [S^n \wedge G/H_+, X]^G \to [S^n \wedge G/gHg_+^{-1}, X]^G \\
&= (\mathrm{id} \wedge (-)g_+^{-1})_* : [S^n, X \wedge G/H_+]^G \to [S^n, X \wedge G/gHg_+^{-1}]^G
\end{aligned}$$

for any subgroups $K \subset H \subset G$ and group element $g \in G$. In general, we let

$$\underline{\pi}_n X(T) = [S^n \wedge T_+, X]^G \cong [S^n, X \wedge T_+]^G$$

for any finite G-set T, and we define more general structure maps as above.

The duality isomorphism factors as a composite

$$[S^n, X \wedge G/H_+]^G \cong [S^n, \mathrm{coind}_H^G \mathrm{res}_H^G X]^G \cong [S^n \wedge G/H_+, X]^G$$

of the Wirthmüller isomorphism and the adjunction. Thus, the transfers on $\underline{\pi}_n X$ are induced by the twisted sums $\Sigma_{G/H}$ described earlier. As a special case of Definition 2.1, note that if E is a G-spectrum and X is a based G-space or a G-spectrum, then the E-homology and E-cohomology groups of X

$$E_*^G X = \pi_*^G (E \wedge X) \quad \text{and} \quad E_G^* X = \pi_{-*}^G F_G(X, E)$$

naturally extend to Mackey functors $\underline{E}_*(X) = \underline{\pi}_*(E \wedge X)$ and $\underline{E}^*(X) = \underline{\pi}_{-*} F_G(X, E)$.

Example 2.2. The 0th stable homotopy groups $\underline{\pi}_0(S^0)$ of the equivariant 0-sphere are isomorphic to the Burnside Mackey functor $\underline{A}$. Explicitly, for any subgroups $K \subset H \subset G$, let $\chi(H/K)$ denote the Euler characteristic

$$S^0 \xrightarrow{\mathrm{coev}} H/K_+ \wedge D(H/K_+) \simeq D(H/K_+) \wedge H/K_+ \xrightarrow{\mathrm{ev}} S^0$$

of H/K_+, regarded as a class in $\pi_0^H S^0$. Then χ induces an isomorphism of Mackey functors $\underline{A} \cong \underline{\pi}_0 S^0$ (cf. [6, V.2]). We may alternatively represent the map $\chi(H/K)$ as the composite

$$S^0 \xrightarrow{\eta} \mathrm{coind}_K^H \mathrm{res}_K^H S^0 \simeq \mathrm{ind}_K^H \mathrm{res}_K^H S^0 \xrightarrow{\varepsilon} S^0.$$

More generally, there is an isomorphism

$$\mathscr{B}_G(X, Y) \cong [\Sigma_+^\infty X, \Sigma_+^\infty Y]^G$$

for any finite G-sets X and Y, which identifies $\mathscr{B}_G$ with the full subcategory $\langle \mathrm{Orb}_G \rangle \subset \mathrm{ho}\,\mathbf{Sp}^G$ spanned by suspensions of finite pointed G-sets [6, V.9]. For this reason, $\mathscr{B}_G$ is also called the *stable orbit category*. For any G-spectrum X and integer $n \in \mathbb{Z}$, taking nth stable homotopy groups defines a functor $[-, \Sigma^{-n} X]^G : \langle \mathrm{Orb}_G \rangle^{\mathrm{op}} \to \mathbf{Ab}$. The isomorphism $\mathscr{B}_G \cong \langle \mathrm{Orb}_G \rangle$ ensures that we recover the Mackey functor structure of Definition 2.1, and nothing more.

3 Spectral Mackey functors

The previous section suggests that G-spectra are categorifications of Mackey functors. We now present a theorem that gives substance to this idea.

Regard a Mackey functor as an additive functor $\underline{M} : \mathscr{B}_G^{\mathrm{op}} \to \mathbf{Ab}$ from the Burnside category to the category of abelian groups (Definition 1.5). Every part of this definition has a higher algebraic counterpart. The idea is to replace $\mathbf{Ab}$ with the category $\mathbf{Sp}$ of nonequivariant spectra, and to enhance $\mathscr{B}_G$ to a spectrally enriched category $\mathscr{B}_{G,sp}$ that models the full, spectral subcategory of $\mathbf{Sp}^G$ spanned by suspensions of finite pointed G-sets. We follow Guillou and May's model categorical treatment [4], where $\mathscr{B}_{G,sp}$ is denoted $G\mathscr{A}$, but Barwick has proven a similar theorem using ∞-categories [1]. An important antecedent to these results appears in earlier work of Schwede and Shipley [9, Example 3.4.(i)]. They show that $\mathbf{Sp}^G$ is equivalent to the category of spectral presheaves over the full subcategory of $\mathbf{Sp}^G$ spanned by $\{\Sigma^\infty_+ G/H \mid H \subset G\}$. A related result for presentable stable ∞-categories is given in [7, Proposition 1.4.4.9].

Remark 3.1. The difference between [9] and the work in [1] and [4] is that the latter papers construct the spectral Burnside category without reference to equivariant homotopy theory. Barwick works over a natural ∞-categorical lift of the Lindner category $\mathscr{B}_G^+$, which he does not group complete. As we explain below, Guillou and May work over a precise spectral analogue to $\mathscr{B}_G$, constructed by homotopy group completing a 2-categorical lift of $\mathscr{B}_G^+$.

Recall that the Lindner 1-category $\mathscr{B}_G^+$ has finite G-sets $X, Y, Z, \ldots$ as objects, and that a morphism from X to Y in $\mathscr{B}_G^+$ is an isomorphism class of a span $X \leftarrow U \to Y$ of finite G-sets. The first step in constructing $\mathscr{B}_{G,sp}$ is to remember the isomorphisms

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] \arrow[dd, "\cong"'] & \\ X & & Y \\ & U' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

between different representatives of morphisms in $\mathscr{B}_G^+$. It will be technically convenient to think of a span $X \leftarrow U \to Y$ as a single morphism $U \to Y \times X$, and to consider G-actions on the finite sets $\varnothing, \{1\}, \{1,2\}, \{1,2,3\}, \ldots$ only. Among other things, this cuts the proper class of finite G-sets down to a countable set, and it makes the disjoint union and cartesian product of G-sets strictly associative and unital. *Henceforth, we understand all finite G-sets to be of this form.*

Definition 3.2. For any finite G-set A, let $G\mathscr{E}(A)$ be the category of finite G-sets and G-isomorphisms over A. This is a strictly associative and unital symmetric monoidal category (also called a *permutative category*) under disjoint union.

Recall that a *bicategory* $\mathscr{C}$ is a category weakly enriched in 1-categories. More explicitly, a bicategory consists of a class of objects $\mathrm{Ob}(\mathscr{C})$, hom 1-categories

$\mathscr{C}(X, Y)$, composition functors $\circ : \mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \to \mathscr{C}(X, Z)$, and identities $\mathrm{id} : * \to \mathscr{C}(X, X)$ such that the usual associative and unital laws hold up to coherent natural isomorphism.

Definition 3.3. Let $G\mathscr{E}$ be the bicategory whose objects are finite G-sets, and whose hom 1-categories are $G\mathscr{E}(X, Y) = G\mathscr{E}(Y \times X)$. Composition corresponds to the pullback of spans, and the diagonal $\Delta : X \to X \times X$ is the identity at X.

The bicategory $G\mathscr{E}$ is very nearly a strict 2-category. Composition is strictly associative, and one of the unit laws holds strictly. We can make the other unit law strict by "whiskering" on new identity elements. To be precise, if $\mathscr{C}$ is a 1-category with basepoint $c \in \mathscr{C}$, then the whiskered category $\mathscr{C}'$ has:

(a) objects $\mathrm{Ob}(\mathscr{C}') = \mathrm{Ob}(\mathscr{C}) \sqcup \{*\}$, and

(b) hom sets $\mathscr{C}'(x, y) = \mathscr{C}(\varepsilon(x), \varepsilon(y))$, where $\varepsilon : \mathrm{Ob}(\mathscr{C}') \to \mathrm{Ob}(\mathscr{C})$ is the identity map on $\mathrm{Ob}(\mathscr{C})$ and sends $*$ to the basepoint $c \in \mathscr{C}$.

Compositions and identities in $\mathscr{C}'$ are inherited from $\mathscr{C}$, and $w = \mathrm{id}_c \in \mathscr{C}'(*, c)$ is a canonical "whisker isomorphism" between $*$ and c in $\mathscr{C}'$. Thus, the inclusion $\mathscr{C} \hookrightarrow \mathscr{C}'$ is an equivalence of categories. We think of $\mathscr{C}'$ as a categorical analogue to the whiskering $X' = X \vee [0, 1]$ of a based space X, but we warn the reader that the classifying space $B(\mathscr{C}')$ is not homeomorphic to $(B\mathscr{C})'$.

Definition 3.4. Let $\mathscr{B}^+_{G,2}$ be the strict 2-category whose objects are finite G-sets $X, Y, Z, \dots$, and whose hom 1-categories are

$$\mathscr{B}^+_{G,2}(X, Y) = \begin{cases} G\mathscr{E}(X, X)' & \text{if } X = Y \text{ and } |X| > 1 \\ G\mathscr{E}(X, Y) & \text{otherwise} \end{cases} .$$

Here we regard $\Delta : X \to X \times X$ as the baspoint of $G\mathscr{E}(X, X)$.

The 2-category $\mathscr{B}^+_{G,2}$ is strictly associative and unital because the whisker isomorphisms provide room to "hang" the bicategorical unit isomorphisms of $G\mathscr{E}$. We refer the reader to [4, §5] for details, where $\mathscr{B}^+_{G,2}$ is denoted $G\mathscr{E}'$.

The homs of $\mathscr{B}^+_{G,2}$ are still permutative categories, and they should be thought of as commutative monoids up to coherent homotopy. It remains to homotopy group complete them. We can do considerably better. Given any permutative category $\mathscr{C}$, there is a connective spectrum $\mathbb{K}\mathscr{C}$ such that $\Omega^\infty\mathbb{K}\mathscr{C}$ is a group completion of the classifying space $B\mathscr{C}$ (cf. [10] and [8]). The basic idea in [10] is to construct the levels of $\mathbb{K}\mathscr{C}$ using a homotopical version of the iterated classifying space construction for topological abelian groups, but nailing down the details is nontrivial. Moreover, the classical versions of $\mathbb{K}\mathscr{C}$ will not suffice for the problem at hand, because producing an honest spectral category $\mathscr{B}_{G,sp}$ from $\mathscr{B}^+_{G,2}$ requires a construction with more precise multiplicative properties, and proving that spectral Mackey functors over $\mathscr{B}_{G,sp}$ are equivalent to G-spectra requires even more compatibilities.

Guillou, May, Merling, and Osorno have developed an "equivariant infinite loop space machine" $\mathbb{K}_G$ with all of the necessary properties in [5] and subsequent work. When $G = e$, the machine $\mathbb{K} = \mathbb{K}_e$

1. sends a permutative category $\mathscr{C}$ to a connective spectrum $\mathbb{K}\mathscr{C}$ whose 0-space is the group completion of $B\mathscr{C}$, and

2. sends multilinear maps between permutative categories to multilinear maps between spectra.

Thus, applying $\mathbb{K}$ to the hom categories of $\mathscr{B}^+_{G,2}$ produces a spectral category.

Definition 3.5. Let $\mathscr{B}_{G,sp}$ be the spectral category whose objects are finite G-sets $X, Y, Z, \dots$, and whose hom spectra are $\mathscr{B}_{G,sp}(X,Y) = \mathbb{K}(\mathscr{B}^+_{G,2}(X,Y))$. A *spectral Mackey functor* is a contravariant spectral functor from $\mathscr{B}_{G,sp}$ to $\mathbf{Sp}$. We write $\mathbf{Mack}_{sp}(G)$ for the category of spectral Mackey functors for G.

Using further properties of $\mathbb{K}_G$, Guillou and May prove that the homotopy theory of spectral G-Mackey functors and the homotopy theory of G-spectra are equivalent.

Theorem 3.6 ([4, Theorem 0.1])**.** *There is a zig-zag of Quillen equivalences connecting $\mathbf{Mack}_{sp}(G)$ to the category $\mathbf{Sp}^G$ of orthogonal G-spectra.*

We briefly indicate some ingredients in the proof. Just as the orbits G/H generate the homotopy theory of $\mathbf{Top}^G$, the suspension spectra $\Sigma^\infty_+ G/H$ generate the homotopy theory of $\mathbf{Sp}^G$. Let $G\mathscr{D}$ denote the full, spectral subcategory of $\mathbf{Sp}^G$ spanned by (bifibrant approximations of) the suspension spectra of finite pointed G-sets. Then the category of spectral contravariant functors from $G\mathscr{D}$ to $\mathbf{Sp}$ is Quillen equivalent to $\mathbf{Sp}^G$. This is essentially Schwede and Shipley's theorem [9]. The rest of the proof boils down to showing that $G\mathscr{D}$ is suitably equivalent to the spectral category $\mathscr{B}_{G,sp}$.

By a non-group complete version of the tom Dieck splitting, the category $\mathscr{B}^+_{G,2}(X,Y)$ is a model for the G-fixed points of $\mathbb{P}_G(Y \times X)_+$, the free G-E_∞ algebra on the based G-set $(Y \times X)_+$. Therefore $\mathscr{B}_{G,sp}(X,Y) \simeq \mathbb{K}(\mathbb{P}_G(Y \times X)^G_+)$, and compatibility relations between $\mathbb{K}$ and $\mathbb{K}_G$ imply that $\mathbb{K}(\mathbb{P}_G(Y \times X)^G_+) \simeq \mathbb{K}_G(\mathbb{P}_G(Y \times X)_+)^G$. From here, the equivariant Barratt-Priddy-Quillen theorem gives $\mathbb{K}_G(\mathbb{P}_G(Y \times X)_+)^G \simeq (\Sigma^\infty_+(Y \times X))^G$, and by duality,

$$(\Sigma^\infty_+(Y \times X))^G \cong (\Sigma^\infty_+ Y \wedge \Sigma^\infty_+ X)^G \simeq F_G(\Sigma^\infty_+ X, \Sigma^\infty_+ Y)^G \simeq F^G(\Sigma^\infty_+ X, \Sigma^\infty_+ Y).$$

Therefore $\mathscr{B}_{G,sp}(X,Y) \simeq G\mathscr{D}(\Sigma^\infty_+ X, \Sigma^\infty_+ Y)$.

Ignoring (co)fibrancy issues, the rest of the proof boils down to checking that these equivalences define an equivalence $\mathscr{B}_{G,sp} \simeq G\mathscr{D}$ of spectral categories, and that they induce an equivalence $\mathbf{Fun}(\mathscr{B}^{\mathrm{op}}_{G,sp}, \mathbf{Sp}) \simeq \mathbf{Fun}(G\mathscr{D}^{\mathrm{op}}, \mathbf{Sp})$ on the level of homotopy theories.

Example 3.7. For any finite G-set X, the representable spectral Mackey functor $\mathscr{B}_{G,sp}(-,X) \in \mathbf{Mack}_{sp}(G)$ and the suspension G-spectrum $\Sigma^\infty_+ X \in \mathbf{Sp}^G$ correspond under the equivalences of Theorem 3.6 (cf. [4, §2.5]). In particular, $\mathscr{B}_{G,sp}(-,G/G)$ correponds to the sphere G-spectrum.

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