Application

Weinan Lin

1 $RO(G)$-Graded Cohomology

Definition 1.1. Let $\mathcal{A}$ be a coefficient system. Then, we say $\tilde{H}^*_G(-,\mathcal{A})$ extends to an $RO(G)$-graded theory if there exist functors

$$\tilde{H}^V_G : hG\mathbf{Top}^{op}_* \to \mathbf{Ab}$$

(where V is a finite dimensional subspace of U) together with isomorphisms

$$\tilde{H}^V_G(X;\mathcal{A}) \cong \tilde{H}^{V\oplus W}_G(\Sigma^W X;\mathcal{A})$$

satisfying the following:

- If V is the n-dimensional trivial representation, $\tilde{H}^V_G(-,\mathcal{A})$ agrees with degree-n Bredon cohomology.
- For each representation V, $\tilde{H}^V_G$ satisfies the additivity, weak equivalence and exactness axioms.

Theorem 1.2 (Lewis-May-McClure, 1981)**.** *Let $\mathcal{A}$ be a coefficient system, and $H^*_G(-,\mathcal{A})$ be the associated ($\mathbb{Z}$-graded) Bredon cohomology theory. Then, $H^*_G(-,\mathcal{A})$ extends to an $RO(G)$-graded theory if and only if $\mathcal{A}$ extends to a Mackey functor.*

We have Brown's representability theorems in the equivariant setting:

Theorem 1.3. *If U is a complete G-universe, then an $RO(G)$-graded cohomology theory on based G-spaces or on G-spectra is representable.*

Theorem 1.4. *A $\mathbb{Z}$-graded cohomology theory on G-spectra indexed on U is represented by a G-spectrum indexed on U and therefore extends to an $RO(G)$-graded cohomology theory on G-spectra indexed on U.*

Remark 1.5. By previous talks we have two models for G-spectra. Therefore we have two interpretations of this theorem.

Theorem 1.6. *For a Mackey functor $\mathcal{M}$, there is an Eilenberg-MacLane spectrum $H\mathcal{M}$ such that $\pi_0(H\mathcal{M}) = \mathcal{M}$ and $\pi_n(H\mathcal{M}) = 0$ if $n \neq 0$. It is unique up to isomorphism in $\bar{h}G\mathbf{Top}_*$. For Mackey functors $\mathcal{M}$ and $\mathcal{M}'$, $[H\mathcal{M}, H\mathcal{M}']_G$ is the group of maps of Mackey functors $\mathcal{M} \to \mathcal{M}'$.*

Proof. To get $H\mathcal{M}$ first we Mimic Bredon's construction of ordinary $\mathbb{Z}$-graded cohomology, but in the category of G-spectra, using Mackey functors instead of coefficient systems. We apply Brown's representability theorem to get $H\mathcal{W}$ such that

$$H_G^V(X;\mathcal{M}) \cong [X, \Sigma^{-V} H\mathcal{M}]_G.$$

Then $H\mathcal{M}$ is the required Eilenberg-MacLane G-spectrum. □

2 Conner conjecture

Theorem 2.1 (Conner conjecture)**.** *Let G be a compact Lie group, and let X have the homotopy type of a finite dimensional G-CW complex with finitely many orbit types. A is an Abelian group. Then,*

$$\tilde{H}^*(X;A) = 0 \implies \tilde{H}^*(X/G;A) = 0.$$

Remark 2.2. If $H \lhd G$, then $X/G \cong (X/H)/(G/H)$.

We need the Oliver transfer map to prove the theorem.

Theorem 2.3 (Oliver Transfer)**.** *Let $\pi : X/H \to X/K$ be the projection map induced by the map on orbits $G/H \to G/K$. This induces a map on cohomology:*

$$\pi^* : \tilde{H}^*(X/K;A) \to \tilde{H}^*(X/H;A).$$

There is a transfer map $\tau : \tilde{H}^*(X/H;A) \to \tilde{H}(X/K;A)$ such that $\tau \circ \pi^*$ is multiplication by the Euler characteristic $\chi(K/H)$.

Proof. We can embed $M = K/H$ into a large G-rep V. Let ν be the normal bundle. We have the following maps

$$t : S^V \to T\nu \to T(\tau_M \oplus \nu) \cong M_+ \wedge S^V$$

It is known that the composite of t and the collapsing map $M_+ \wedge S^V \to S^V$ is the Euler characteristic $\chi(K/H)$.

Note that

$$\tilde{H}^n(X/H;A) \cong \tilde{H}^n_H(X;\underline{A}) \cong \tilde{H}^n_K(X \wedge K/H_+;\underline{A}) \cong \tilde{H}^{n+V}_K(X \wedge \Sigma^V K/H_+;\underline{A})$$

$$\tilde{H}^n(X/K;A) \cong \tilde{H}^n_K(X;\underline{A}) \cong \tilde{H}^{n+V}_K(\Sigma^V X;\underline{A}) \cong \tilde{H}^{n+V}_K(X \wedge S^V;\underline{A})$$

Smashing with X, t induces τ. □

Using Smith theory, we will first prove the Conner conjecture when G is a finite group. If $G = S^1$, we will use the finite case to reduce to the case where the action of G is the free away from the basepoint, and apply a localization result. Then, by the observation, we will have proved the conjecture whenever G can be written as a finite group extension of a torus.

Lemma 2.4. *If G is finite, the statement of the Conner conjecture holds.*

Proof. First, we assume G is a p-group and $A = \mathbb{F}_p$.

By Theorem 2.5 in my previous talk, if X is acyclic, then

$$\sum_q H^q(X^G) \leqslant \sum_q \dim H^q(X) = 1$$

$$\chi(X^G) \equiv \chi(X) = 1 \mod p$$

So X^G is acyclic. Using the inequality of Betti numbers in the proof of Theorem 2.5 (previous talk) we have

$$c_q + i_q + 2\sum_{i=q}^{r} b_i \leqslant 2\sum_{i=q}^{r} a_i$$

where r is sufficiently large that higher degree cohomology is trivial. With $q > 0$, we have $\sum_{i=q}^{r} b_i = \sum_{i=q}^{r} a_i = 0$ so $c_q + i_q = 0 \implies c_q = 0$; if $q = 0$ we have $\sum_{i=0}^{q} b_i = \sum_{i=0}^{r} a_i = 1$ so $c_0 + i_0 = 0 \implies c_0 = 0$. Thus $(X/X^G)/G$ is acyclic. This is enough to show that X/G is acyclic as well because we have the following cofiber sequence:

$$X^G \hookrightarrow X/G \to (X/X^G)/G.$$

Now, let G be an arbitrary finite group and still $A = \mathbb{F}_p$. Let P be a p-Sylow subgroup of G. There is a covering map $p : X/P \to X/G$. Using the Oliver transfer map τ, we have the following composition:

$$\tilde{H}^*(X/G; A) \xrightarrow{p^*} \tilde{H}^*(X/P; A) \xrightarrow{\tau} \tilde{H}^*(X/G; A)$$

This composition is multiplication by $|G/P|$, which is an isomorphism since it is prime to p. But $\tilde{H}^*(X/P; A) = 0$ by the above. So $\tilde{H}^*(X/G; A) = 0$.

Next, let $A = \mathbb{Q}$. We have the composition:

$$\tilde{H}^*(X/G; A) \xrightarrow{p^*} \tilde{H}^*(X; A) \xrightarrow{\tau} \tilde{H}^*(X/G; A)$$

This composition is multiplication by $|G|$, which is an isomorphism, so $\tilde{H}^*(X/G; A) = 0$.

Finally by the universal coefficient theorem we can extend to arbitrary coefficients. □

Lemma 2.5. *If $G = S^1$, the statement of the Conner conjecture holds.*

Proof. All proper subgroups of G are cyclic. Thus, by the finite orbit type hypothesis, there is a large cyclic group C such that C contains all proper stabilizers. Hence X/C is G/C-semifree and acyclic by the previous lemma. We may assume X is a semi-free G-complex. Considering the C_p-subgroup action on X, we have that $X^G = X^{C_p}$, so by Smith theory X^G is $\mathbb{F}_p$-acyclic.

To see that X^G is $\mathbb{Q}$-acyclic, we cite a version of Smith theory from [3]:

Lemma 2.6. *Let X be a semi-free finite dimensional S^1-CW complex. Then, for $i = 0, 1$:*

$$\sum_{q \equiv i \bmod 2} \dim H^q(X^G; \mathbb{Q}) \leqslant \sum_{q \equiv i \bmod 2} \dim H^q(X; \mathbb{Q})$$

Now we have that both X and X^G are acyclic, we can use Vietoris mapping theorem and Serre sequence of the following diagram:

$$\begin{tikzcd}
X^G \arrow[r] \arrow[d] & X \arrow[d] \\
(EG \times X^G)/G \arrow[r, "\phi"] \arrow[d] & (EG \times X)/G \arrow[d] \\
BG \arrow[r, equal] & BG
\end{tikzcd}$$

to get

$$H^*(X/G, X^G) \cong H^*((EG \times X)/G, (EG \times X^G)/G) = 0$$

Hence X/G is acyclic. $\square$

Corollary 2.7. *If G is a finite extension of a torus, the statement of the Conner conjecture holds.*

Lemma 2.8. *Let G be a connected compact Lie group, and T be a maximal torus in G. Then the Euler characteristic $\chi(G/N_G(T)) = 1$.*

Now we can prove the Conner conjecture:

Proof of Theorem 2.1. Let G_0 be the component of the identity in G; this is a normal subgroup since conjugation is a continuous map fixing the identity. Let T be the maximal torus of G_0 and N be its normalizer. We know that N/T is finite. By Corollary 2.7, $\tilde{H}^*(X/N; A) = 0$. We have the composition

$$\tilde{H}^*(X/G_0; A) \xrightarrow{\pi^*} \tilde{H}^*(X/N; A) \xrightarrow{\tau} \tilde{H}^*(X/G_0; A)$$

where τ is the Oliver transfer. By Lemma 2.8, this composition is the identity, so we have $\tilde{H}^*(X/G_0; A) = 0$. Since G is a finite extension of G_0, we have $\tilde{H}^*(X/G; A) = 0$. $\square$

References

[1] J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996, With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. MR1413302

[2] R. Oliver, *A proof of the Conner conjecture*, Ann. of Math. (2) **103** (1976), no. 3, 637–644. MR0415650

[3] P. Conner, *Retraction Properties of the Orbit Space of a Compact Topological Transformation Group*. Duke Mathematical Journal 27.3 (1960): 341-57. Print.