# Equivariant $K$-theory 

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## 1 Classical Equivariant K-theory

### 1.1 Equivariant vector bundles

Definition 1.1. Let $G$ be a compact Lie group and $X$ be a $G$-manifold. A $G$-vector bundle over $X$ is a $G$-map $\xi: E \rightarrow X$ which is a vector bundle such that $G$ acts linearly on the fibers, i.e. $g: E_{X} \rightarrow E_{g x}$ is linear for all $g \in G$ and $x \in X$.

If $X$ is a compact space, $K_{G}(X)$ is defined to the Grothendieck group of finite dimensional complex $G$-vector bundles over $X$. Tensor product of $G$-vector bundle makes $K_{G}(X)$ into a ring.

Example 1.2. $K_{G}(\mathrm{pt}) \simeq R(G)$ is the representation ring of $G$.
A $G$-bundle is trivial if it is trivial as a vector bundle. By Peter-Weyl Theorem, any $G$-vector bundle over $X$ is a summand of a trivial $G$-bundle. $\widetilde{K}_{G}(X):=$ $\operatorname{Coker}\left(K_{G}(\mathrm{pt}) \rightarrow K_{G}(X)\right)$ is the stable isomorphism classes of $G$-bundles over $X$.

Proposition 1.3. If $X$ is $G$-free, then $K_{G}(X) \simeq K(X / G)$. If $G$ acts trivially on $X$, then $K_{G}(X) \simeq R(G) \otimes K(X)$.

Let $i: H \hookrightarrow G$ be an inclusion of a subgroup. For a $G$-space $X$, there is a restriction map $i^{*}: K_{G}(X) \rightarrow K_{H}(X)$. On the other hand, if $H$ is of finite index in $G$, there is a transfer map induced by the induction of representations $i_{*}: K_{H}(X) \rightarrow$ $K_{G}(X)$. When $H$ is of infinite index in $G$, this may lead to vector bundles of infinite dimensions. In this case, one may hope to use three other constructions: smooth induction, holomorphic transfer and dimension-shifting transfer for $R O(G)$-graded theories.

For a locally compact space $X$, define $K_{G}(X):=\widetilde{K}_{G}\left(X_{+}\right)$.
Theorem 1.4 (Thom isomorphism). For vector bundles $E$ over a locally compact space $X$, there is a natural Thom isomorphism:

$$
\phi: K_{G}(X) \xrightarrow{\sim} K_{G}(E) .
$$

Reducing to the case when $X$ is a point and $E=V$ for some $V \in R(G)$. Define $\lambda(V) \in R(G)$ to be

$$
\lambda(V)=1-V+\Lambda^{2} V-\cdots+(-1)^{\operatorname{dim} V} \Lambda^{\operatorname{dim} V} V
$$

Let $e_{V}: S^{0} \rightarrow S^{V}$ be the based map sending the non-basepoint to 0 and let $b_{V}=\phi(1) \in \widetilde{K}\left(S^{V}\right)$.

Theorem 1.5 (Equivariant Bott periodicity). For a compact $G$-space $X$ and a complex $G$-representation $V$, multiplication by $b_{V}$ specifies a an isomorphism

$$
\phi: \widetilde{K}_{G}\left(X_{+}\right)=K_{G}(X) \xrightarrow{\sim} K_{G}(V \times X)=\widetilde{K}_{G}\left(S^{V} \wedge X_{+}\right) .
$$

Moreover, $e_{V}^{*}\left(b_{V}\right)=\lambda(V)$.
The proof of the Thom isomorphism for line bundles can be proved using clutching functions as in the non-equivariant case. This implies the theorem for $G$ abelian since every $G$-representation is a direction sum of one-dimensional representations. The proof of the $G=U(n)$ case needs holomorphic transfer. For $K O_{G}$, the Bott periodicity theorem holds for spin representations whose dimension is divisible by 8 . The proof needs pseudo-differential operators.

Now we can extend $K_{G}(-)$ to a reduced cohomology theory by setting in degree zero $K_{G}^{0}\left(X_{+}\right)=K_{G}(X)$ for finite $G$-CW complexes $X$ and $K_{G}^{0}(X)=\widetilde{K}_{G}(X)$ for based finite $G-C W$ complexes $X$. Bott periodicity implies we may take:

$$
K_{G}^{2 n}(X):=K_{G}^{0}(X) \text { and } K_{G}^{2 n+1}(X):=K_{G}^{0}\left(\Sigma^{1} X\right) \text { for all } n
$$

Let $I=I(G)=\operatorname{ker}(\operatorname{dim}: R(G) \rightarrow \mathbb{Z})$ be the augmentation. For $G$-space $X$, there is a free $G$-map $\pi: E G_{+} \wedge_{G} X \rightarrow X$.

Theorem 1.6 (The Atiyah-Segal completion theorem). [EHCT, XIV.5] The map $\pi^{*}: K_{G}^{*}(X) \rightarrow K_{G}^{*}\left(E G_{+} \wedge_{G} X\right)$ is the completion at the augmentation ideal. In particular, $K_{G}^{0}\left(E G_{+}\right) \simeq R(G) \wedge$ and $K_{G}^{1}\left(E G_{+}\right)=0$.

Remark 1.7. Recall that for any based space $X, X^{h G}=\operatorname{map}\left(E G_{+}, X\right)^{G}$. In this sense, the Borel equivariant $K$-theory $K_{G}^{*}\left(E G_{+} \wedge-\right)$ corresponds to the homotopy fixed points and $K_{G}^{*}(-)$ corresponds to fixed points.

### 1.2 Atiyah's $K R$-theory

Definition 1.8. A space with involution is a space $X$ with a homeomoprhism $\tau$ : $X \rightarrow X$ of period 2. $\tau(x)$ is also written as $\bar{x}$. A point $x \in X$ is a real point if $x=\bar{x}$. The subspace of $X$ fixed by $\tau$ is denoted by $X_{\mathbb{R}}$.

A real vector bundle over a real space $X$ is a complex vector bundle $\xi: E \rightarrow X$ such that

1. $E$ is a real space.
2. $\xi$ is $\mathbb{Z} / 2$-equivariant.
3. The map $E_{x} \rightarrow E_{\bar{x}}$ is complex anti-linear.

The Grothendieck groups of the real vector bundles over a real space $X$ is denoted by $K R(X)$.

Remark 1.9. A real vector bundle is different from a $\mathbb{Z} / 2$-vector bundle. The map $E_{x} \rightarrow E_{\bar{x}}$ is complex-linear for the latter.

Proposition 1.10. Let $X$ be a real space with $X=X_{\mathbb{R}}$, then there is a natural equivalence between the category of real vector bundles over $X$ as space and the category of real vector bundles over $X$ as real space. The equivalence maps a vector bundle $E$ to $E \otimes_{\mathbb{R}} \mathbb{C}$. As a result, $K R(X) \simeq K O(X)$.

We are now going to extend $K R$ into an $R O(\mathbb{Z} / 2)$-graded theory. Denote by $\mathbb{R}$ the real space with trivial involution and $i \mathbb{R}$ the real space with involution $i x \rightarrow-i x$. We use the following notation:

$$
\mathbb{R}^{p, q}:=\mathbb{R}^{q} \oplus i \mathbb{R}^{\oplus p}, \quad B^{p, q}:=\text { unit ball in } \mathbb{R}^{p, q}, \quad S^{p, q}:=\text { unit sphere in } \mathbb{R}^{p, q}
$$

Notice that $\mathbb{R}^{p, p} \simeq \mathbb{C}^{p}$ as real spaces and $S^{p, q}$ has dimension $p+q-1$. Define $\widetilde{K R}$ to be kernel of restriction to the base point and set $K R(X, Y):=\widetilde{K R}(X / Y)$. Now define

$$
K R^{p, q}(X, Y):=K R\left(X \times B^{p, q}, X \times S^{p, q} \cup Y \times B^{p, q}\right)
$$

Then the usual suspension are given by $K R^{-q}=K R^{0, q}$. The $K R$-periodicity is described as follows. Let

$$
b=[H]-1 \in K R^{1,1}(\mathrm{pt})=K R\left(B^{1,1}, S^{1,1}\right)=K R\left(\mathbb{C P}^{1}\right)
$$

where $H$ is the tautological line bundle over $\mathbb{C P}^{1}$.
Theorem 1.11 ( $K R$-periodicity). $\beta: K R^{p, q}(X, Y) \xrightarrow{\sim} K R^{p+1, q+1}(X, Y), x \mapsto b \cdot x$ is an isomorphism.

Remark 1.12. The $K R$-periodicity is related to a similar periodicity of representations of Clifford algebras: $M_{2}(\mathbb{R}) \otimes C \ell_{p, q} \simeq C \ell_{p+1, q+1}$

We can recover the Bott periodicity for $K$ and $K O$ from the $K R$-periodicity.
Proposition 1.13. There are natural isomorphisms:

$$
\begin{aligned}
& K R\left(S^{1,2} \times X\right) \simeq K R\left(S^{1,0} \times X\right) \simeq K(X) \\
& K R\left(S^{2,4} \times X\right) \simeq K R\left(S^{2,0} \times X\right) \simeq K S C(X) \\
& K R\left(S^{4,8} \times X\right) \simeq K R\left(S^{4,0} \times X\right)
\end{aligned}
$$

KSC is the self-conjugate $K$-theory.

## 2 Global Equivariant K-theory Spectra

### 2.1 Borel equivariant theories from non-equivariant spectra

Let $E$ be a non-equivariant generalized cohomology theory. We obtain a global functor:

$$
\underline{E}(G):=E^{0}(B G) .
$$

The contravariant functoriality of $\underline{E}$ in $G$ follows from the functoriality of classifying spaces. The transfer map for inclusions of subgroups $H \subset G$ is the Becker-Gottlieb transfer tr : $\Sigma_{+}^{\infty} B G \rightarrow \Sigma_{+}^{\infty} B H$. More generally, for a compact Lie group $G$ and a cofibrant $G$-space $X$, we have the Borel equivariant cohomology theory represented by $E: E^{*}\left(E G \times_{G} X\right)$. These Borel cohomology theories are represented by a global homotopy type.
Construction 2.1. GHT 4.5.21] Given an orthogonal spectrum $E$, we define a new orthogonal spectrum $b E$ by setting

$$
(b E)(V)=\operatorname{Map}\left(\mathbf{L}\left(V, \mathbb{R}^{\infty}\right), E(V)\right)
$$

for an inner product space $V . O(V)$ acts on $b E(V)$ by conjugation. The structure maps $\sigma_{V, W}$ are defined by the composites

$$
\begin{aligned}
& S^{V} \wedge \operatorname{Map}\left(\mathbf{L}\left(W, \mathbb{R}^{\infty}\right), E(W)\right) \xrightarrow{\text { assembly }} \operatorname{Map}\left(\mathbf{L}\left(W, \mathbb{R}^{\infty}\right), S^{V} \wedge E(W)\right) \\
& \xrightarrow{\text { Map }\left(\operatorname{res}_{W}, \sigma_{V, W}^{E}\right)} \operatorname{Map}\left(\mathbf{L}\left(V \oplus W, \mathbb{R}^{\infty}\right), E(V \oplus W)\right),
\end{aligned}
$$

where $\operatorname{res}_{W}: \mathbf{L}\left(V \oplus W, \mathbb{R}^{\infty}\right) \rightarrow \mathbf{L}\left(W, \mathbb{R}^{\infty}\right)$ is the restriction of isometric embeddings. The structure maps are functorially given by

$$
\begin{aligned}
\mathbf{O}(V, W) \wedge \operatorname{Map}\left(\mathbf{L}\left(V, \mathbb{R}^{\infty}\right), E(V)\right) & \longrightarrow \operatorname{Map}\left(\mathbf{L}\left(W, \mathbb{R}^{\infty}\right), E(W)\right) \\
(w, \varphi) & \wedge f
\end{aligned}>\{\psi \mapsto E(w, \varphi)(f(\psi \circ \varphi))\}
$$

The endofunctor $b$ on orthogonal spectra comes a natural transformation $i_{E}$ : $E \rightarrow b E$ that send $s$ a points $x \in E(V)$ to the constant map $\mathbf{L}\left(V, \mathbb{R}^{\infty}\right) \rightarrow E(V)$ with value $x$. This morphism is a non-equivariant level equivalence as $\mathbf{L}\left(V, \mathbb{R}^{\infty}\right)$ is contractible.

We endow the functor $b$ with a lax symmetric monoidal transformation

$$
\mu_{E, F}: b E \wedge b F \longrightarrow b(E \wedge F)
$$

$\mu_{E, F}$ is constructed using the universal property of the smash product. It suffices to define the $O(V) \times O(W)$-equivariant maps that constitute a bimorphism from $(b F, b E)$ to $b(E \wedge F)$ :

$$
\begin{aligned}
& \operatorname{Map}\left(\mathbf{L}\left(V, \mathbb{R}^{\infty}\right), E(V)\right) \wedge \operatorname{Map}\left(\mathbf{L}\left(W, \mathbb{R}^{\infty}\right), F(W)\right) \\
& \xrightarrow{\wedge} \operatorname{Map}\left(\mathbf{L}\left(V, \mathbb{R}^{\infty}\right) \times \mathbf{L}\left(W, \mathbb{R}^{\infty}\right), E(V) \wedge F(W)\right) \\
& \xrightarrow{\operatorname{map}\left(\text { res }_{V, W, i v, W)}\right.} \operatorname{Map}\left(\mathbf{L}\left(V \oplus W, \mathbb{R}^{\infty}\right),(E \wedge F)(V \oplus W)\right),
\end{aligned}
$$

where $\operatorname{res}_{V, W}: \mathbf{L}\left(V \oplus W, \mathbb{R}^{\infty}\right) \rightarrow \mathbf{L}\left(V, \mathbb{R}^{\infty}\right) \times \mathbf{L}\left(W, \mathbb{R}^{\infty}\right)$ maps the embedding of $V \oplus W$ to its restriction to $V$ and $W$.

### 2.2 Connective global K-theory

Definition 2.2. Let $\Gamma$ be the category whose objects are based finite sets $n_{+}=$ $\{0, \cdots, n\}$ with base point 0 and whose morphisms are based maps. A $\Gamma$-space is a functor from $\boldsymbol{\Gamma}$ to the category of based spaces that is reduced, i.e. the value at $0_{+}$is a one-point space.

Construction 2.3. Let $\mathcal{U}$ be a complex vector space of countable dimension, equipped with a hermitian inner product. For a finite based set $A$, let $C(\mathcal{U}, A)$ be the space of tuples $\left(E_{a}\right)$, indexed by non-base point elements of $A$ of finite-dimensional, pairwise orthogonal $\mathbb{C}$-subspaces of $\mathcal{U}$. The topology on $C(\mathcal{C}, A) \subseteq \prod_{A} \operatorname{Gr}(\mathcal{U})$ is the subspace topology. The base point of $C(\mathcal{U}, A)$ is the tuple with $E_{a}=0$ for all a. A based map $\alpha: A \rightarrow B$ induces a map $C(\mathcal{U}, A) \rightarrow C(\mathcal{U}, B)$ sending $\left(E_{a}\right)$ to $\left(F_{b}\right)$, where

$$
F_{b}=\bigoplus_{\alpha(a)=b} E_{a}
$$

Then $C(\mathcal{U})$ is a $\Gamma$-space whose underlying space is

$$
C\left(\mathcal{U}, 1_{+}\right)=\coprod_{n \geqslant 0} \operatorname{Gr}_{n}^{\mathbb{C}}(\mathcal{U}) .
$$

Every $\Gamma$-space can be evaluated on a based space by the coend construction. Write $C(\mathcal{U}, K)=C(\mathcal{U})(K)$. Elements of $C(\mathcal{U}, K)$ are represented by an unordered tuple:

$$
\left[E_{1}, \cdots, E_{n} ; k_{1}, \cdots, k_{n}\right]
$$

where $\left(E_{1}, \cdots, E_{n}\right)$ is an $n$-tuple of finite-dimensional, pairwise orthogonal subspaces of $\mathcal{U}$ and $k_{1}, \cdots, k_{n}$ are points of $K$.
Definition 2.4. A $C^{\star}$-algebra $A$ is a Banach algebra over $\mathbb{C}$ with a map $x \rightarrow x^{\star}$ for $x \in A$ such that

- $(-)^{\star}$ is an involution, i.e $\left(x^{\star}\right)^{\star}=x$.
- $(x+y)^{\star}=x^{\star}+y^{\star}$ and $(x y)^{\star}=y^{\star} x^{\star}$.
- For $\lambda \in \mathbb{C},(\lambda x)^{\star}=\bar{\lambda} x^{\star}$.
- $\left\|x x^{\star}\right\|=\|x\|\left\|x^{\star}\right\|$.

A bounded linear map $f: A \rightarrow B$ of $C^{\star}$-algebras is a $\star$-homomorphism if $f(x y)=$ $f(x) f(y)$ and $f\left(x^{\star}\right)=\left(f(x)^{\star}\right)$. $f$ is called $\star$-isomomorphism if it is bijective. In that case, say $A$ and $B$ are isomorphic.
Remark 2.5. When $K$ is a finite based discrete space, $C(\mathcal{U}, K)$ agrees with the original definition. When $K$ is compact, we have

$$
C(\mathcal{U}, K) \simeq \operatorname{colim}_{V \subset \mathcal{U}, \operatorname{dim} V<\infty} C^{\star}\left(C_{0}(K), \operatorname{End}_{\mathbb{C}}(V)\right)
$$

where $C_{0}(K)$ is the $C^{\star}$-algebra of continuous $\mathbb{C}$-valued functions on $K$ that vanish at the base point.
Remark 2.6. $C\left(\mathcal{U}, S^{1}\right) \simeq U(\mathcal{U})$, the group of unitary isometries on $\mathcal{U}$ that is identity on the orthogonal complement of some finite dimensional subspace of $\mathcal{U}$.

The symmetric power of a vector space $V$ is defined to be $\operatorname{Sym}^{n}(V)=V^{\otimes n} / \Sigma_{n}$. Write $\operatorname{Sym}(V)=\oplus_{n \geqslant 0} \operatorname{Sym}^{n}(V)$. This is the free commutative $\mathbb{C}$-algebra generated by $V$. As $\mathbb{C}$-algebras, $\operatorname{Sym}(V) \otimes \operatorname{Sym}(W) \simeq \operatorname{Sym}(V \oplus W)$.

Lemma 2.7. If $V$ has a Hermitian inner product, then there is a preferred inner product on $\operatorname{Sym}^{n}(V)$, making $\operatorname{Sym}(V) \otimes \operatorname{Sym}(W) \simeq \operatorname{Sym}(V \oplus W)$ an isometry.

Construction 2.8. [GHT, 6.3.9] We define the connective global $K$-theory spectrum ku by setting

$$
\mathbf{k u}(V)=C\left(\operatorname{Sym}\left(V_{\mathbb{C}}\right), S^{V}\right)
$$

where $V$ is an Euclidean inner product space, $V_{\mathbb{C}}$ is its complexification with the induced Hermitian inner product. The action of $O(V)$ on $V$ extends to a unitary action on $\operatorname{Sym}\left(V_{\mathbb{C}}\right)$. $O(V)$ then acts on $C\left(\operatorname{Sym}\left(V_{\mathbb{C}}\right), S^{V}\right)$ diagonally.

To define the ring spectrum structure on $\mathbf{k u}$, first notice for based spaces $K$ and $L$, there is a continous multiplication map:

$$
\begin{aligned}
C(\mathcal{U}, K) \wedge C(\mathcal{V}, L) & \longrightarrow C(\mathcal{U} \otimes \mathcal{V}, K \wedge L) \\
{\left[E_{i} ; k_{i}\right] \wedge\left[F_{j} ; I_{j}\right] } & \longmapsto\left[E_{i} \otimes F_{j} ; k_{i} \wedge I_{j}\right]
\end{aligned}
$$

These multiplication maps are associative and commutative. Now define a an $(O(V) \times O(W))$-equivariant multiplication map

$$
\begin{aligned}
\mu_{V, W}: \mathbf{k u}(V) \wedge \mathbf{k u}(W) & =C\left(\operatorname{Sym}\left(V_{\mathbb{C}}\right), S^{V}\right) \wedge C\left(\operatorname{Sym}\left(W_{\mathbb{C}}\right), S^{W}\right) \\
& \rightarrow C\left(\operatorname{Sym}\left(V_{\mathbb{C}}\right) \otimes \operatorname{Sym}\left(W_{\mathbb{C}}\right), S^{V} \wedge S^{W}\right) \\
& \simeq C\left(\operatorname{Sym}\left((V \oplus W)_{\mathbb{C}}\right), S^{V \oplus W}\right)=\mathbf{k u}(V \oplus W)
\end{aligned}
$$

The maps $\mu_{V, W}$ are associative and commutative. The $O(V)$-equivariant unit map is given by

$$
\iota v: S^{V} \longrightarrow C\left(\operatorname{Sym}\left(V_{\mathbb{C}}\right), S^{V}\right)=\mathbf{k u}(V), \quad v \mapsto[\mathbb{C} \cdot 1 ; v]
$$

where $\mathbb{C} \cdot 1:=\operatorname{Sym}^{0}\left(V_{\mathbb{C}}\right)$.
Construction 2.9 (Complex conjugation on $\mathbf{k u}$ ). $\operatorname{As~} \operatorname{Sym}\left(V_{\mathbb{C}}\right) \simeq \mathbb{C} \otimes_{\mathbb{R}} \operatorname{Sym}_{\mathbb{R}}(V)$, $\operatorname{Sym}\left(V_{\mathbb{C}}\right)$ has an involution $\psi_{\operatorname{Sym}(V)}$, inducing the complex conjugation on $\mathbf{k u}$ :

$$
\psi_{V}=C\left(\psi_{\operatorname{Sym}(V)}, S^{V}\right): \mathbf{k u}(V) \longrightarrow \mathbf{k u}(V)
$$

Construction 2.10. Analogously, one can define the connective real global Ktheory by setting

$$
\mathbf{k o}(V)=C_{\mathbb{R}}\left(\operatorname{Sym}(V), S^{V}\right)
$$

where $C_{\mathbb{R}}(\mathcal{U})$ is the $\Gamma$-space of tuples of pairwise orthogonal, finite-dimensional real subspaces of $\mathcal{U}$.

Proposition 2.11. [GHT Proposition 6.3.17] The orthogonal spectra ku and ko are globally connective.

### 2.3 Periodic global K-theory

Construction 2.12 (The $C^{\star}$-algebra s). A graded $C^{\star}$-algebra is a $C^{\star}$-algebra $A$ equipped with a $\star$-automorphism $\alpha: A \rightarrow A$ such that $\alpha^{2}=$ id. The $\pm 1$-eigenspaces of $\alpha$ give a $\mathbb{Z} / 2$-grading on $A$ :

$$
A_{\text {even }}=\{a \in A \mid \alpha(a)=a\} \quad \text { and } \quad A_{\text {odd }}=\{a \in A \mid \alpha(a)=-a\}
$$

The conjugation of $A$ preserves this grading.
Let $s$ be the $C^{\star}$-algebra of complex valued continuous functions on $\mathbb{R}$ vanishing at $\infty$. The involution on $s$ is given by

$$
\alpha(f)(t)=f(-t)
$$

The even and odd parts of $s$ under this involution coincide with the usual definition of even and odd functions.

Construction 2.13 (Complex Clifford algebras). Let $V$ be a Euclidean inner product space. Define the complex Clifford algebra $\mathbb{C} \ell(V)$ to be

$$
\mathbb{C} \ell(V):=(T V)_{\mathbb{C}} /\left(v \otimes v-|v|^{2} \cdot 1\right)
$$

where $T V=\oplus_{n \geqslant 0} V^{\otimes n}$ is the free tensor algebra on $V$. The original space $V$ embeds into the degree 1 part of $T V$ and thus the odd part of $\mathbb{C} \ell(V)$. This construction is functorial for $\mathbb{R}$-linear isometric embedding, so the action of $O(V)$ on $V$ naturally extends to $\mathbb{C} \ell(V)$.
$\mathbb{C} \ell(V)$ is a $\mathbb{Z} / 2$-graded $O(V)$ - $C^{\star}$-algebra. The involution on $\mathbb{C} \ell(V)$ is defined by setting $[v]^{\star}=[v]$ for $v \in V \subseteq \mathbb{C} \ell(V)$ and extending this to a $\mathbb{C}$-semilinear anti-automorphism. The norm on $\mathbb{C} \ell(V)$ is a restriction of operator norms via an embedding $\mathbb{C} \ell(V) \hookrightarrow \operatorname{End}_{\mathbb{C}}\left(\Lambda^{*}\left(V_{\mathbb{C}}\right)\right)$. This embedding is an extension of $V \hookrightarrow$ End $_{\mathbb{C}}\left(\Lambda^{*}\left(V_{\mathbb{C}}\right)\right)$ by the universal property of $\mathbb{C} \ell(V)$.

The $\mathbb{C} \ell$ functor is monoidal, i.e there is an isomorphism of graded $\mathbb{C}$-algebras

$$
\mu_{V, W}: \mathbb{C} \ell(V \oplus W) \xrightarrow{\sim} \mathbb{C} \ell(V) \otimes \mathbb{C} \ell(W),
$$

where $\otimes$ is the graded tensor product.
Construction 2.14 (Periodic global K-theory). [GHT, 6.4.9] Let $V$ be a Euclidean inner product space. $\operatorname{Sym}\left(V_{\mathbb{C}}\right)$ inherits a Hermtian inner product and an $O(V)$ action by $\mathbb{C}$-linear isometries. $\operatorname{Sym}\left(V_{\mathbb{C}}\right)$ is infinite dimensional (unless $V=0$ ) but not complete. Denote by $\mathcal{H}_{V}$ the Hilbert space completion of $\operatorname{Sym}\left(V_{\mathbb{C}}\right)$. The action $O(V)$ extends to $\mathcal{H}_{V}$, making it a complex Hilbert space representation of $O(V)$. Denote by $\mathcal{K}_{V}$ the $C^{\star}$-algebra of compact operators on $\mathcal{H}_{V}$. Now define the periodic global K-theory KU by setting

$$
\mathbf{K U}(V)=C_{\operatorname{gr}}^{\star}\left(s, \mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right)
$$

Here, we consider $\mathcal{K}_{V}$ as evenly graded, so the grading of $\mathbb{C} \ell(V) \otimes \mathcal{K}_{V}$ is entirely from $\mathbb{C} \ell(V)$. The topology of $\mathbf{K U}(V)$ comes from the pointwise convergence topology in the operator norm of $\mathcal{K}_{V}$ and the base point is the zero $\star$-homomorphism.

The continuous action of $O(V)$ on $\mathcal{H}_{V}$ induces an $O(V)$-action on $\mathcal{K}_{V}$ by conjugation. Together with the $O(V)$-action on $\mathbb{C} \ell(V)$, it gives an $O(V)$-action on $\operatorname{KU}(V)$.

The multiplication on $\mathbf{K U}$ is defined as follows. First there is isomorphism of $C^{\star}$-algebras

$$
\mathcal{K}_{V} \hat{\otimes} \mathcal{K}_{W} \simeq \mathcal{K}_{V \oplus W},
$$

where $\hat{\otimes}$ is the complete tensor product. From this we obtain an isomorphism of graded $C^{\star}$-algebras:

$$
\begin{aligned}
\left(\mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right) \hat{\otimes}\left(\mathbb{C} \ell(W) \otimes \mathcal{K}_{W}\right) & \simeq(\mathbb{C} \ell(V) \otimes \mathbb{C} \ell(W)) \otimes\left(\mathcal{K}_{V} \hat{\otimes}_{\mathcal{K}_{W}}\right) \\
& \simeq \mathbb{C} \ell(V \oplus W) \otimes \mathcal{K}_{V \oplus W}
\end{aligned}
$$

Now define the multiplication map $\mu_{V, W}: \mathbf{K U}(V) \wedge \mathbf{K U}(W) \rightarrow \mathbf{K U}(V \oplus W)$ as the composite:

$$
\begin{aligned}
C_{\mathrm{gr}}^{\star}\left(s, \mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right) \wedge C_{\mathrm{gr}}^{\star}\left(s, \mathbb{C} \ell(W) \otimes \mathcal{K}_{W}\right) & \xrightarrow{\hat{\otimes}} C_{\mathrm{gr}}^{\star}\left(s \hat{\otimes} s,\left(\mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right) \hat{\otimes}\left(\mathbb{C} \ell(W) \otimes \mathcal{K}_{W}\right)\right) \\
& \xrightarrow{\Delta^{*}} C_{\mathrm{gr}}^{\star}\left(s,\left(\mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right) \hat{\otimes}\left(\mathbb{C} \ell(W) \otimes \mathcal{K}_{W}\right)\right) \\
& \xrightarrow{\sim} C_{\mathrm{gr}}^{\star}\left(s, \mathbb{C} \ell(V \oplus W) \otimes \mathcal{K}_{V \oplus W}\right) .
\end{aligned}
$$

These multiplication maps are associative and commutative. To describe the unit map of this multiplication, we need to define the "functional calculus" map:

$$
\mathrm{fc}: S^{V} \longrightarrow C_{\mathrm{gr}}^{\star}\left(s, \mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right), v \mapsto(-)[v]
$$

where for $f \in s$

$$
f[v]=\mathrm{fc}(v)(f)=\left\{\begin{array}{cl}
f(|v|) \cdot 1, & \text { when } f \text { is even; } \\
\frac{f(|v|)}{|v|} \cdot[v] & \text { when } f \text { is odd and } v \neq 0 \\
0 & \text { when } f \text { is odd and } v=0
\end{array}\right.
$$

Define the unit map $\eta_{V}: S^{V} \rightarrow C_{\mathrm{gr}}^{\star}\left(s, \mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right)$ by

$$
S^{V} \xrightarrow{\mathrm{fc}} C_{\mathrm{gr}}^{\star}\left(s, \mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right) \xrightarrow{\left(-\otimes p_{0}\right)_{*}} C_{\mathrm{gr}}^{\star}\left(s, \mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right)=\mathbf{K U}(V),
$$

where $p_{0} \in \mathcal{K}_{V}$ is the projection of the symmetric algebra onto its 0 -th summand.
Construction 2.15. We now describe a ring spectrum map $j: \mathbf{k u} \rightarrow \mathbf{K U}$. For each inner product space $V$, we need to define

$$
j(V): \mathbf{k u}(V)=C\left(\operatorname{Sym}\left(V_{\mathbb{C}}\right), S^{V}\right) \longrightarrow C_{\mathrm{gr}}^{\star}\left(s, \mathbb{C} \ell(V) \otimes \mathcal{K}_{V}\right)=\mathbf{K U}(V)
$$

For a configuration

$$
\left[E_{1}, \cdots, E_{n} ; v_{1}, \cdots, v_{n}\right] \in C\left(\operatorname{Sym}\left(V_{\mathbb{C}}\right), S^{V}\right)
$$

we define the associated $\star$-homomorphism to be

$$
\begin{aligned}
j(V)\left[E_{1}, \cdots, E_{n} ; v_{1}, \cdots, v_{n}\right]: s & \longrightarrow \mathbb{C} \ell(V) \otimes \mathcal{K}_{V} \\
f & \longmapsto \sum_{i=1}^{n} f\left[v_{i}\right] \otimes p_{E_{i}}
\end{aligned}
$$

where $f\left[v_{i}\right]=\mathrm{fc}\left(v_{i}\right)(f)$ and $p_{E}$ denotes the orthogonal projection onto a subspace $E$. One can check $j(V)\left[E_{k} ; v_{k}\right]$ is $\mathbb{Z} / 2$-graded. Furthermore, $j(V)$ is $O(V)$-equivariant, multiplicative and unital. Thus the maps $j(V)$ indeed form a morphism of ultracommutative ring spectra.

We now describe the equivariant Bott periodicity for KU. Let $G$ be a compact Lie group, and $V$ an orthogonal $G$-representation. Denote by $C_{0}(V, \mathbb{C} \ell(V))$ the $G-C^{\star}$-algebra of continuous $\mathbb{C} \ell(V)$-valued functions on $V$ that vanish at infinity. Functional calculus provides a distinguished graded $\star$-homomorphism:

$$
\beta_{V}: s \longrightarrow C_{0}(V, \mathbb{C} \ell(V)), \quad \beta_{V}(f)(v)=f[v] .
$$

As the functional calculus map is $G$-equivariant, $\beta_{V}$ is invariant under the conjugation action of $G$ on $C_{0}(V, \mathbb{C} \ell(V))$. Let $\mathcal{H}_{G}$ be any complete $G$-Hilbert space universe and $\mathcal{K}_{G}$ be the $G-C^{\star}$-algebra of (not necessarily equivariant) compact operators on $\mathcal{H}_{G}$, with $G$ acting by conjugation.

Theorem 2.16 (Equivariant Bott Periodicity). [GHT, Theorem 6.4.17] For every G-C*-algebra A, the map

$$
\beta_{V} \cdot-: C_{\mathrm{gr}}^{\star}\left(s, A \otimes \mathcal{K}_{G}\right) \longrightarrow C_{\mathrm{gr}}^{\star}\left(s, C_{0}(V, \mathbb{C} \ell(V)) \otimes A \otimes \mathcal{K}_{G}\right)
$$

is a $G$-weak equivalence.
Construction 2.17. We define global connective $K$-theory $\mathbf{k u}^{c}$. This will NOT be a connective equivariant theory as the homotopy groups $\pi_{*}^{G}\left(\mathbf{k u}^{c}\right)$ do not vanish in negative degrees if $G$ is non-trivial. This is a "globalization" of the connective equivariant $K$-theory in $[\mathrm{Gr}] . \mathbf{k u}^{c}$ is the homotopy pullback of the following diagram:


Here, $b(-)$ is the Borel construction in 2.1 that comes with an endofunctor $i:$ id $\rightarrow$ $b$, and $j: \mathbf{k u} \rightarrow \mathbf{K U}$ is the spectrum map constructed in 2.15 .

## References

[Ati] M. F. Atiyah, K-theory and reality, Quart. J. Math. Oxford Ser. (2) 17 (1966), 367-386. MR0206940
[Gr] J. P. C. Greenlees, Equivariant connective K-theory for compact Lie groups, J. Pure Appl. Algebra 187 (2004), no. 1-3, 129-152. MR2027899
[EHCT] J. P. May, Equivariant homotopy and cohomology theory, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996, With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. MR1413302
[GHT] S. Schwede, Global homotopy theory, New Mathematical Monographs, vol. 34, Cambridge University Press, Cambridge, 2018. MR3838307

