Equivariant cobordism

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1 Equivariant Thom-Pontryagin construction

Goal: Explain the following diagram of equivariant homology theories:

$$\begin{tikzcd}
\mathcal{N}_*^G \arrow[r, "\Theta^G"] \arrow[d] & \mathbf{mO}_*^G \arrow[r] \arrow[d] & \mathbf{mOP}_*^G \arrow[d] \\
\mathfrak{N}_*^{G:S} \arrow[r, "\cong", "\Theta^G"'] & \mathbf{MO}_*^G \arrow[r] & \mathbf{MOP}_*^G
\end{tikzcd}$$

In this diagram:

1. **MO** is the ultra-commutative Thom spectrum, **mO** is an E_∞-Thom spectrum. **MOP** and **mOP** are periodic extensions of **MO** and **mO**, respectively.

2. The vertical transformation in the middle column is an isomorphism for $G = e$. This is not true in general.

3. $\mathcal{N}_*^G$ is a geometrically defined equivariant bordism and $\mathfrak{N}_*^{G:S}$ is a *stable* equivariant bordism. They are not represented by orthogonal spectra, but defined from bordism classes of G-manifolds.

4. The two Θ^G maps are equivariant Thom-Pontryagin construction and its stabilization". The upper Θ^G is an isomorphism when G is a product of finite groups and a torus.

1.1 Global Thom spectra

We first define the global Thom spectra: **MGr**, **MOP**, **MO**, **mOP** and **mO**.

Example 1.1. We start with **MGr**, the Thom spectrum over the additive Grassmannian **Gr**. The value of **Gr** at each inner product space V is

$$\mathbf{Gr}(V) = \coprod_{n\geqslant 0} Gr_n(V).$$

The total space of the tautological Euclidean vector bundle (of no constant rank) over $\mathbf{Gr}(V)$ consists of points (x, U) such that $x \in U \in \mathbf{Gr}(V)$. We define $\mathbf{MGr}(V)$

to be the Thom space of this tautological bundle over $\mathbf{Gr}(V)$. The structure maps are given by

$$\begin{aligned}\mathbf{O}(V,W)\wedge\mathbf{MGr}(V)&\longrightarrow\mathbf{MGr}(W)\\(w,\varphi)\wedge(x,U)&\longmapsto(w+\varphi(x),\varphi^{\perp}\oplus\varphi(U)),\end{aligned}$$

where $\varphi^{\perp}$ is the orthogonal complement of the image of $\varphi: V\to W$. Multiplication maps are defined by direct sum:

$$\begin{aligned}\mu_{V,W}:\mathbf{MGr}(V)\wedge\mathbf{MGr}(W)&\longrightarrow\mathbf{MGr}(V\oplus W)\\(x,U)\wedge(x',U')&\longmapsto((x,x'),(U,U')).\end{aligned}$$

Unit maps are defined by

$$\eta(V):S^V\longrightarrow\mathbf{MGr}(V),\quad v\longmapsto(v,V).$$

The multiplication maps are binatural, associative, commutative and unital, making $\mathbf{MGr}$ an ultra-commutative ring spectrum. $\mathbf{MGr}$ is graded, with the k-th homogeneous summand given by

$$\mathbf{MGr}^{[k]}(V)=Th\left(Gr_{|V|+k}(V)\right).$$

This shows $\mathbf{MGr}$ is concentrated in non-positive degrees and the unit morphism $\eta:\mathbb{S}\to\mathbf{MGr}$ is an isomorphism onto $\mathbf{MGr}^{[0]}$. Let V be a representation of a compact Lie group G, we define the *inverse Thom class* $\tau_{G,V}\in\mathbf{MGr}_0^G(S^V)$ as a class represented by the G-map:

$$\begin{aligned}t_{G,V}:S^V&\longrightarrow Th(Gr(V))\wedge S^V=\mathbf{MGr}(V)\wedge S^V\\v&\longmapsto(0,\{0\})\wedge(-v).\end{aligned}$$

The internal degree of $\tau_{G,V}$ is equal to $-\dim V$.

Example 1.2. We define two ultra-commutative ring spectra $\mathbf{MO}$ and $\mathbf{MOP}$. $\mathbf{MOP}$ is a Thom spectra over the orthogonal space $\mathbf{BOP}$, whose value at an inner product space V is

$$\mathbf{BOP}(V)=\coprod_{n\geqslant 0}Gr_n(V^2).$$

Define $\mathbf{MOP}(V)$ as the Thom space of the tautological vector bundle over $\mathbf{BOP}(V)$. The structure maps are given by

$$\begin{aligned}\mathbf{O}(V,W)\wedge\mathbf{MOP}(V)&\longrightarrow\mathbf{MOP}(W)\\(w,\varphi)\wedge(x,U)&\longmapsto((w,0)+\mathbf{BOP}(\varphi)(x),\mathbf{BOP}(\varphi)(U)).\end{aligned}$$

Multiplication maps are defined by

$$\begin{aligned}\mu_{V,W}:\mathbf{MOP}(V)\wedge\mathbf{MOP}(W)&\longrightarrow\mathbf{MOP}(V\oplus W)\\(x,U)\wedge(x',U')&\longmapsto(\kappa_{V,W}(x,x'),\kappa_{V,W}(U\oplus U')),\end{aligned}$$

where $\kappa_{V,W} V^2 \oplus W^2 \xrightarrow{\sim} (V \oplus W)^2$ is the preferred isometry defined by

$$\kappa_{V,W}((v,v'),(w,w')) = ((v,w),(v',w')).$$

Unit maps are defined by

$$\eta^V : S^V \longrightarrow \mathbf{MOP}(V), \quad v \longmapsto ((v,0),(V \oplus 0)).$$

The multiplication maps make **MOP** an ultra-commutative ring spectrum. The orthogonal space **BOP** is $\mathbb{Z}$-graded, with k-th homogeneous summand

$$\mathbf{BOP}^{[k]}(V) = Gr_{|V|+k}(V^2).$$

The Thom spectra **MOP** inherits the $\mathbb{Z}$-grading from **BOP**. $\mathbf{MOP}(V)$ is the wedge sum of the Thom spaces $\mathbf{MOP}^{[k]}(V)$ for $-|V| \leqslant k \leqslant |V|$ and thus

$$\mathbf{MOP} = \bigvee_{k\in\mathbb{Z}} \mathbf{MOP}^{[k]}.$$

We define $\mathbf{MO} = \mathbf{MOP}^{[0]}$. It is an ultra-commutative ring spectrum on its own right. Explicitly, $\mathbf{MO}(V)$ is the Thom space of the tautological vector bundle over $Gr_{|V|}(V^2)$.

Example 1.3. We define two E_∞-ring spectra **mO** and **mOP**, the Thom spectra over the orthogonal spaces **bO** and **bOP**. The value of **bOP** at an inner product space V is

$$\mathbf{bOP}(V) = \coprod_{n\geqslant 0} Gr_n(V \oplus \mathbb{R}^\infty),$$

For a linear isometric embedding $\varphi : V \to W$, the induced map $\mathbf{bOP}(\varphi) : \mathbf{bOP}(V) \to \mathbf{bOP}(W)$ is defined as

$$\mathbf{bOP}(\varphi)(L) = (\varphi \oplus \mathbb{R}^\infty)(L) + ((W - \varphi(V)) \oplus 0).$$

Over $\mathbf{bOP}(V)$ sits a tautological Euclidean vector bundle (of non-constant rank) and we define $\mathbf{mOP}(V)$ as the Thom space of this tautological bundle. The structure maps are given by

$$\begin{aligned} \mathbf{O}(V,W) \wedge \mathbf{mOP}(V) &\longrightarrow \mathbf{mOP}(W) \\ (w,\varphi) \wedge (x,U) &\longmapsto ((w,0) + \mathbf{bOP}(\varphi)(x), \mathbf{bOP}(\varphi)(U)). \end{aligned}$$

The E_∞-structures on **mO** and **mOP** are inherited from those on **bO** and **bOP** by the linear isometry operad. Multiplication maps are defined by

$$\begin{aligned} \mu_{V,W} : \mathbf{L} \wedge \mathbf{mOP}(V) \wedge \mathbf{mOP}(W) &\longrightarrow \mathbf{mOP}(V \oplus W) \\ \psi \wedge (x,U) \wedge (x',U') &\longmapsto (\psi_\sharp(x,x'), \psi_\sharp(U \oplus U')), \end{aligned}$$

where $\psi_\sharp$ is the linear isometric embedding

$$\begin{aligned} \psi_\sharp : V \oplus \mathbb{R}^\infty \oplus W\mathbb{R}^\infty &\longrightarrow V \oplus W \oplus \mathbb{R}^\infty \\ (v,y,w,z) &\longmapsto (v,w,\psi(y,z)). \end{aligned}$$

Unit maps are defined by

$$\eta^V : S^V \longrightarrow \mathbf{mOP}(V), \quad v \longmapsto ((v, 0), (V \oplus 0)).$$

$\mathbf{mOP}$ is $\mathbb{Z}$-graded, $\mathbf{mOP}^{[k]}(V)$ is the Thom space of the tautological bundle over $\mathbf{bOP}^{[k]}(V) = Gr_{|V|+k}(V \oplus \mathbb{R}^\infty)$. Then $\mathbf{mOP}(V)$ is the wedge sum of $\mathbf{mOP}^{[k]}(V)$ for $|V| + k \geqslant 0$ and there is a decomposition

$$\mathbf{mOP} = \bigvee_{k\in\mathbb{Z}} \mathbf{mOP}^{[k]}.$$

We define $\mathbf{mO} = \mathbf{mOP}^{[0]}$ to the zeroth summand in this decomposition.

The equivariant cohomology theories represented by the global Thom spectra are related by the following:

$$\mathbf{MGr}^G_*(A) \xRightarrow{\text{invert } \tau_{G,\mathbb{R}}} \mathbf{mOP}^G_*(A) \xRightarrow{\text{invert all } \tau_{G,V}} \mathbf{MOP}^G_*(A)$$

More precisely, we define maps $a : \mathbf{MGr} \to \mathbf{MOP}$ and $b : \mathbf{MGr} \to \mathbf{mOP}$, whose values at an inner product space V are

$$\begin{aligned} a(V) &: \mathbf{MGr}(V) \longrightarrow \mathbf{MOP}(V) & (x, L) &\longmapsto ((x, 0), L \oplus 0), \\ b(V) &: \mathbf{MGr}(V) \longrightarrow \mathbf{mOP}(V) & (x, L) &\longmapsto ((x, 0), L \oplus 0). \end{aligned}$$

The localized $\mathbf{MOP}$ and $\mathbf{mOP}$ are defined by

$$\begin{aligned} \mathbf{MGr}^G_k(A)[1/\tau_{G,V}] &= \operatorname*{colim}_{V\in s(\mathcal{U}_G)} \mathbf{MGr}^G_k(A \wedge S^V) \\ \mathbf{MGr}^G_k(A)[1/\tau_{G,\mathbb{R}}] &= \operatorname*{colim}_{n\geqslant 0} \mathbf{MGr}^G_k(A \wedge S^n), \end{aligned}$$

where the structure maps are

$$\begin{aligned} \mathbf{MGr}^G_k(A \wedge S^V) \xrightarrow{\tau_{G,W-V}} \mathbf{MGr}^G_k(A \wedge S^V \wedge S^{W-V}) &\simeq \mathbf{MGr}^G_k(A \wedge S^W), \\ \mathbf{MGr}^G_k(A \wedge S^n) \xrightarrow{\tau_{G,\mathbb{R}}} \mathbf{MGr}^G_k(A \wedge S^n \wedge S^{\mathbb{R}}) &\simeq \mathbf{MGr}^G_k(A \wedge S^{n+1}). \end{aligned}$$

Theorem 1.4. *The maps a and b are compatible with the colimits and they assemble into maps*

$$\begin{aligned} a^\sharp &: \mathbf{MGr}^G_k(A)[1/\tau_{G,V}] \longrightarrow \mathbf{MOP}^G_k(A), \\ b^\sharp &: \mathbf{MGr}^G_k(A)[1/\tau_{G,\mathbb{R}}] \longrightarrow \mathbf{mOP}^G_k(A). \end{aligned}$$

The maps $a^\sharp$ and $b^\sharp$ are isomorphisms for every compact Lie group G, based G-space A and integer k.

1.2 Geometric equivariant bordism

Definition 1.5. Let G be a compact Lie group and X be a G-space. A *singular G-manifold over X* is a pair (M, h), where M is a closed smooth G-manifold and $h : M \to X$ is a continuous G-map. Two singular G-manifolds (M, h) and (M', h') are *bordant* if there is a triple (B, H, ψ), where B is a compact smooth G-manifold, $H : B \to X$ is continuous G-map, and ψ is an equivariant diffeomorphism:

$$\psi : M \cup M' \xrightarrow{\sim} \partial B$$

such that $(H \circ \psi)|_M = h$ and $(H \circ \psi)|_{M'} = h'$.

Bordism of singular G-manifolds over X is an equivalence relation. We denote by $\mathcal{N}_n^G(X)$ the set of bordism classes of n-dimensional singular G-manifolds over X. The sets becomes an abelian group under disjoint union.

Proposition 1.6. *$\mathcal{N}_*^G$ is an equivariant homology theory, i.e. it satisfies the following:*

1. *Functorial in continuous G-maps.*
2. *G-equivariant homotopy invariant.*
3. *Takes G-weak equivalences to isomorphisms.*
4. *Takes disjoint unions of G-spaces to direct sums.*
5. *Has Mayer-Vietoris sequences for good pairs of G-spaces.*

Construction 1.7. There is a distinguished class $d_{G,V} \in \widetilde{\mathcal{N}}_*^G(S^V)$ for a G-representation V. Stereographic projection is a G-equivariant map

$$\Pi_V : S(\mathbb{R} \oplus V) \xrightarrow{\sim} S^V, \quad (x, v) \longmapsto \frac{v}{1-x}.$$

We define a reduced G-bordism class over S^V by

$$d_{G,V} = [\![S(\mathbb{R} \oplus V), \Pi_V]\!] \in \widetilde{\mathcal{N}}_{|V|}^G(S^V).$$

Proposition 1.8. *$d_{G,V} \wedge d_{G,W} = d_{G,V\oplus W} \in \widetilde{\mathcal{N}}_{|V|+|W|}^G(S^{V\oplus W})$.*

If G acts trivially on V and $X$is a cofirant based G-space, then the exterior product map with $d_{G,V}$ is an isomorphism:

$$- \wedge d_{G,V} : \widetilde{\mathcal{N}}_n^G(X) \xrightarrow{\sim} \widetilde{\mathcal{N}}_{n+|V|}^G(X \wedge S^V).$$

Construction 1.9. To every smooth closed G-manifold M, we assoicate a normal class $\langle M \rangle \in \mathbf{MGr}_0^G(M_+)$. This class is the geometric input for the Thom-Pontryagin map to equivariant $\mathbf{mO}$-homology. If $\dim M = m$, then the class lives in the summand $\mathbf{MGr}^{[-m]}$ of $\mathbf{MGr}$.

By Mostow-Palais embedding theorem, there is a G-equivariant embedding $i : M \hookrightarrow V$ for some G-representation V. Without loss of generality, assume V is a sub-representation of the chosen complete G-universe $\mathcal{U}_G$. Define ν to be the normal bundle of the embedding, where the metric is provided by the inner product on V. We can also assume, the embedding is *wide* in the sense that the exponential map $(x, v) \mapsto i(x) + v$ on the unit disk bundle $D(\nu)$ of ν is a G-embedding into a tubular neighborhood of M. This determines a G-equivariant Thom-Pontryagin map

$$c_M : S^V \longrightarrow Th(Gr(V)) \wedge M_+ = \mathbf{MGr}(V) \wedge M_+$$

by sending points outside the tubular neighborhood to the base point and

$$c_M(i(x) + v) = \left(\frac{v}{1 - |v|}, \nu_x \right) \wedge x.$$

The normal class is the homotopy class of the collapse map c_M.

Proposition 1.10. *The normal class does not depend on the choice of a wide embedding.*

Construction 1.11. Equivariant Thom-Pontryagin construction:

$$\Theta^G = \Theta^G(X) : \widetilde{\mathcal{N}}_*^G(X) \longrightarrow \mathbf{mO}_*^G(X).$$

Let (M, h) be an m-dimensional singular G-manifold over a based G-space X. All the geometry is encoded in the normal class $\langle M \rangle \in \mathbf{MGr}_0^G(M_+)$. We define

$$\Theta^G[M, h] = (b \wedge h)_* \langle M \rangle \cdot p_G^*(\sigma^m) \in \mathbf{mO}_m^G(X),$$

where $b : \mathbf{MGr} \to \mathbf{mOP}$ is a map of ring spectra whosed value at an inner product space V is

$$b(V) : \mathbf{MGr}(V) \to \mathbf{mOP}(V), \quad (x, L) \mapsto ((x, 0), (L \oplus 0)),$$

$\sigma \in \pi_1^e(\mathbf{mOP}^{[1]})$ is periodicity class, inverse to $t \in \pi_{-1}^e(\mathbf{mOP}^{[-1]})$ represented by

$$(0, \{0\}) \in Th(Gr_0(\mathbb{R} \oplus \mathbb{R}^\infty)) = \mathbf{mOP}^{[-1]}(\mathbb{R}),$$

and $p_G : G \to e$ is the projection map that induces a map $p_G^* : \pi_*^e(-) \to \pi_*^G(-)$.

Proposition 1.12. *The class $\Theta^G[M, h] \in \mathbf{mO}_m^G(X)$ only depends on the bordism class of the singular G-manifold (M, h).*

Example 1.13. $\Theta^G(d_{G,V}) = \overline{\tau}_{G,V} \in \mathbf{mO}_m^G(S^V)$ is the shifed inverse Thom class in $\mathbf{mO}$.

Theorem 1.14. *Θ^G is a transformation of equivariant homology theories and compatible with homomorphisms of compact Lie groups.*

Theorem 1.15 (Wasserman)**.** *Let G be a compact Lie group that is isomorphic to a product of finite group and a torus . Then for every cofibrant G-space X, the Thom-Pontryagin map*

$$\Theta^G(X) : \mathcal{N}_*^G(X) \longrightarrow \mathbf{mO}_*^G(X_+)$$

is an isomorphism.

Construction 1.16. We define stable equivariant bordism groups $\widetilde{\mathfrak{N}}^{G:S}_*(X)$ of a based G-space X as the localization of $\widetilde{\mathcal{N}}^G_*(X)$ by formally inverting all classes $d_{G,V}$. That is

$$\widetilde{\mathfrak{N}}^{G:S}_*(X) = \operatorname*{colim}_{V\in s(\mathcal{U}_G)} \widetilde{\mathcal{N}}^G_{m+|V|}(X \wedge S^V),$$

where $s(\mathcal{U}_G)$ is the poset of finite dimensional G-representations in the G-universe $\mathcal{U}_G$ and for $V \subseteq W$ the structure map in the colimit is the multiplication

$$\widetilde{\mathcal{N}}^G_{m+|V|}(X \wedge S^V) \xrightarrow{-\wedge d_{G,W-V}} \widetilde{\mathcal{N}}^G_{m+|W|}(X \wedge S^V \wedge S^{W-V}) \cong \widetilde{\mathcal{N}}^G_{m+|W|}(X \wedge S^W).$$

As the Thom-Pontryagin construction takes $d_{G,V}$ to the shifed inverse Thom class $\overline{\tau}_{G,V} \in \mathbf{mO}_{|}V|^G(S^V)$, the following diagram commutes

$$\begin{tikzcd}
\widetilde{\mathcal{N}}^G_m(X) \arrow[r, "\Theta^G"] \arrow[d, "-\wedge d_{G,V}"'] & \mathbf{mO}^G_m(X) \arrow[d, "-\cdot\overline{\tau}_{G,V}"] \\
\widetilde{\mathcal{N}}^G_{m+|V|}(X \wedge S^V) \arrow[r, "\Theta^G"] & \mathbf{mO}^G_{m+|V|}(X \wedge S^V).
\end{tikzcd}$$

The colimit of this diagram assembles into a natural transformation:

$$\Theta^G : \widetilde{\mathfrak{N}}^{G:S}_m(X) \longrightarrow \mathbf{mO}^G_m(X)[1/\tau].$$

Theorem 1.17. *For every compact Lie group G and every cofibrant based G-space X, then map*

$$\Theta^G(X) : \widetilde{\mathfrak{N}}^{G:S}_*(X) \longrightarrow \mathbf{mO}^G_*(X)[1/\tau]$$

is an isomorphism of graded abelian groups.

Corollary 1.18. *For a cofirant based G-space, there are natural isomorphisms:*

$$\widetilde{\mathfrak{N}}^{G:S}_m(X) \xrightarrow{\Theta^G} \mathbf{mO}^G_m(X)[1/\tau] \xrightarrow{\sim} \mathbf{MO}^G_*(X).$$

2 Equivariant complex cobordism spectra

2.1 Complex cobordism and formal groups

Definition 2.1. A cohomology theory is called **complex oriented** if it is multiplicative and it satisfies Thom isomorphism for (almost) complex vector bundles.

Proposition 2.2. *Let E be a complex oriented cohomology theory, then*

1. $E^*(\mathbb{CP}^\infty) \simeq E_*[\![t]\!]$ *where $t \in E^2(\mathbb{CP}^\infty)$ is the first Chern class of the tautological line bundle ξ over $\mathbb{CP}^\infty$.*
2. *Let $p_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ be the projection map of the i-th component for $i = 1, 2$. Then $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq E_*[\![t_1, t_2]\!]$, where $t_i = p_i^* c_1(\xi)$.*

3. *The tensor product of line bundles over $\mathbb{CP}^\infty$ induces a E_0-formal group structure on $\mathrm{spf}\, E(\mathbb{CP}^\infty)$. Denote this formal group associated to a complex-oriented cohomology theory E by $\widehat{G}_E$.*

4. *$E(S^{2k})$ can be identified ω^k, the k-th tensor power of the sheaf of invariant differentials on $\widehat{G}_E$.*

Example 2.3. Here are two examples of complex oriented cohomology theories and their associated formal groups:

1. For ordinary cohomology theory, $\widehat{G}_H \simeq \widehat{G}_a$ is the additive formal group.

2. For complex K-theory, $\widehat{G}_K \simeq \widehat{G}_m$ is the multiplicative formal group.

Theorem 2.4 (Quillen)**.** *The formal group associated to periodic complex cobordism MUP is the universal formal group. More precisely, the pair*

$$(MU_*, MU_*(MU)) = (MUP_0, MUP_0(MUP))$$

classifies formal groups and strict isomorphisms between formal groups.

2.2 Real bordism

Let ρ_2 be the real regular representation of C_2.

Construction 2.5. We construct the real cobordism spectrum $MU_{\mathbb{R}}$. It is a C_2-equivariant commutative ring admitting a canonical homotopy presentation

$$MU_{\mathbb{R}} \simeq ho\varinjlim S^{-\mathbb{C}^n} \wedge MU(n) \simeq ho\varinjlim S^{-n\rho_2} \wedge MU(n).$$

We will first construct a commutative real algebra $\mathcal{MU}_{\mathbb{R}} \in \mathbf{CAlg}(\mathbf{Sp}_{\mathbb{R}})$ and apply the Quillen equivalence:

$$i_! : \mathbf{Sp}_{\mathbb{R}} \xleftrightarrow{\quad} \mathbf{Sp}^{C_2} : i^* \ .$$

We define $MU_{\mathbb{R}}$ to be the spectrum $i_! \mathcal{MU}'_{\mathbb{R}}$, where $\mathcal{MU}'_{\mathbb{R}} \to \mathcal{MU}_{\mathbb{R}}$ is a cofibrant commutative algebra approximation. Elements in this construction are described below:

Definition 2.6. The category $I_{\mathbb{C}}$ is the topological category whose objects are finite dimensional Hermitian vector spaces and whose morphism space is the Thom space

$$I_{\mathbb{C}}(A, B) = Th(U(A, B); B - A),$$

where $U(A, B)$ is the Stiefel manifold of unitary embeddings $A \hookrightarrow B$ and $B - A$ is the orthogonal completement of A in B under the embedding.

The category The category $I_{\mathbb{R}}$ is the C_2-equivariant topological category whose objects are finite dimensional orthogonal real vector spaces and whose morphism space is the Thom space

$$I_{\mathbb{R}}(V, W) = I_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}}),$$

with C_2 acting by complex conjugation.

Definition 2.7. The category $\mathbf{Sp}_{\mathbb{C}}$ of complex spectra is the topological category of (continuous) functors $I_{\mathbb{C}} \to \mathcal{T}$.

The category $\mathbf{Sp}_{\mathbb{R}}$ of *real spectra* is the topological category of C_2-enriched functors $I_{\mathbb{R}} \to \underline{\mathcal{T}}_{C_2}$ and equivariant natural transformations.

Let $i : I_{\mathbb{R}} \to I_{C_2}$ be the functor sending V to $V \otimes \rho_2$. The restriction functor $i^* : \mathbf{Sp}^{C_2} \to \mathbf{Sp}_{\mathbb{R}}$ has both a left and right adjoint denoted by $i_!$ and i_*, respectively. $i_!$ sends $S^{-V_{\mathbb{C}}}$ to $S^{-V_{\rho_2}}$.

We define the real spectrum $\mathcal{MU}_{\mathbb{R}}$ by sending $V \in I_{\mathbb{R}}$ to $MU(V_{\mathbb{C}})\, Th(BU(V_{\mathbb{C}}), V_{\mathbb{C}})$ with C_2 acting by complex conjugation. $\mathcal{MU}_{\mathbb{R}} \in \mathbf{CAlg}(\mathbf{Sp}_{\mathbb{R}})$ as the functor is a lax symmetric monoidal if we use Segal's construction of $BU(V_{\mathbb{C}})$.

Proposition 2.8.

1. *The non-equivariant spectrum underlying $MU_{\mathbb{R}}$ is the usual complex cobordism spectrum MU.*
2. *There is a equivalence $\Phi^{C_2} MU_{\mathbb{R}} \simeq MO$.*

We now describe the relations between $MU_{\mathbb{R}}$, real orientations and formal groups. Consider $\mathbb{CP}^n$ and $\mathbb{CP}^\infty$ as pointed C_2-spaces under complex conjugation, with $\mathbb{CP}^0$ the base point. The fixed point spaces are $\mathbb{RP}^n$ and $\mathbb{RP}^\infty$, and ther are homeomorphisms $\mathbb{CP}^n/\mathbb{CP}^{n-1} \simeq S^{n\rho_2}$. In particular $\mathbb{CP}^1 \simeq S^{\rho_2}$.

Definition 2.9 (Araki)**.** Let E be C_2-equivariant homotopy commutative ring spectrum. A real orientation of E is a class $\overline{x} \in \widetilde{E}^{\rho_2}_{C_2}(\mathbb{CP}^\infty)$ whose restriction to

$$\widetilde{E}^{\rho_2}_{C_2}(\mathbb{CP}^1) = \widetilde{E}^{\rho_2}_{C_2}(S^{\rho_2}) \simeq E^0_{C_2}(pt)$$

is a unit. A real oriented spectrum is a C_2-equivariant ring spectrum E equipped with a real orientation.

Example 2.10. The zero section $\mathbb{CP}^\infty \to MU(1)$ is an equivariant equivalence and defines a real orientation

$$\overline{x} \in MU^{\rho_2}_{\mathbb{R}}(\mathbb{CP}^\infty),$$

making $MU_{\mathbb{R}}$ into a real oriented spectrum.

Example 2.11. If $(X, \overline{x}_H)$ and $(E, \overline{x}_E)$ are two real oriented spectra, then $H \wedge E$ has two real orientations given by $\overline{x}_H \otimes 1$ and $1 \otimes \overline{x}_E$.

Theorem 2.12 (Araki)**.** *Let E be a real oriented cohomology theory, then there are isomorphisms*

$$E^\star(\mathbb{CP}^\infty) \simeq E^\star[\![\overline{x}]\!],$$
$$E^\star(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq E^\star[\![\overline{x} \otimes 1, 1 \otimes \overline{x}]\!].$$

It follows the tensor product map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ defines a formal group law over $\pi^G_\star$. A real orientation $\overline{x}$ corresponds to a coordinate the corresponding formal group.

If $(E, \overline{x}_E)$ is a real oriented spectrum, then $E \wedge MU_{\mathbb{R}}$ has two orientations $\overline{x}_E = \overline{x}_E \otimes 1$ and $\overline{x}_{\mathbb{R}} = 1 \otimes \overline{x}$. These two series are related by a power series

$$\overline{x}_{\mathbb{R}} = \sum \overline{b}_i \overline{x}_E^{i+1},$$

that defines classes

$$\overline{b}_i = \overline{b}_i^E \in \pi_{i\rho_2}^{C_2} E \wedge MU_{\mathbb{R}}.$$

This power series is an isomorphism of formal group laws F_E to $F_{\mathbb{R}}$ over $\pi_\star^{C_2} E \wedge MU_{\mathbb{R}}$, where F_E and $F_{\mathbb{R}}$ are formal groups associated to $(E, \overline{x}_E)$ and $(MU_{\mathbb{R}}, \overline{x}_{\mathbb{R}})$, respectively.

Theorem 2.13 (Araki)**.** *The map*

$$E_\star[\overline{b}_1, \overline{b}_2, \cdots] \to \pi_\star^{C_2} E \wedge MU_{\mathbb{R}}$$

is an isomorphism.

Passing to geometric fixed points

$$\overline{x} : \mathbb{CP}^\infty \to \Sigma^{\rho_2} MU_{\mathbb{R}} \xrightarrow{\text{geom fixed pt}} a : \mathbb{RP}^\infty \simeq MO(1) \to \Sigma MO$$

defines the MO Euler class of the tautological line bundle. Like MU_*, Quillen shows that the multiplication $\mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ induces a formal group law over MO_* that is universal formal group law F over a ring of characteristic 2 such that $[2]_F = 0$.

Let $e \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2)$ be the $H\mathbb{Z}/2$ Euler class (Stiefel-Whitney class) of the tautological line bundle. Over $\pi_*(H\mathbb{Z}/2 \wedge MO)$, the classes e and a are related by a power series

$$e = \ell(a) = a + \sum \alpha_n a^{n+1}.$$

Lemma 2.14. *The composite series*

$$\left(a + \sum \alpha_{2^j-1} a^{2^j}\right)^{-1} \circ \ell(a) = a + \sum_{j>0} h_j a^{j+1}$$

has coefficients in $\pi_* MO$. The classes $h_{2^k-1} = 0$ and the remaining h_j are polynomial generators for the unoriented cobordism ring:

$$\pi_* MO = \mathbb{Z}/2[h_j \mid j \neq 2^k - 1].$$

Let $G = C_{2^n}$ and localize all spectra at the prime 2. Write $g = |G|$ and let $\gamma \in G$ be a fixed generator.

Definition 2.15. $MU^{((G))} := N_{C_2}^G MU_{\mathbb{R}}$

For $H \subset G$, the unit of the restriction-norm adjunction gives a canonical commutative algebra map

$$MU^{((H))} \to i_H^* MU^{((G))}.$$

Write i_1^* for $i_{C_2}^*$.

2.3 Universal properties of real bordism

Let R_* be a graded ring and $F(x,y) \in R_*[\![x,y]\!]$ be a homogeneous formal group ($\deg x = \deg y = -2$). Let $c : R_* \to R_*$ be a graded ring homomorpshism such that $c_{2n} : R_{2n} \to R_{2n}$ is multiplication by $(-1)^n$. Define $F^c = c^* F$, we have

$$F^c(x,y) = -F(-x,-y).$$

c induces strict isomorphisms $F \xrightarrow{\sim} F^c$ and $F^c \xrightarrow{\sim} F$ by $c(x) = -[-1]_F(x)$. This is called the conjuate action on F.

Proposition 2.16. *[HHR, Example 11.27] $MU_{\mathbb{R}}$ is universal in the sense that $MU_* \to R_*$ classifying a homogeneous formal group law is C_2-equivariant for any choice of conjugation action.*

The real orientation $i_1^* MU_{\mathbb{R}} \to MU^{((G))}$ for $G = C_{2^n}$ induces a formal group law F with a G-action that extends the conjugation action on by $C_2 \subseteq G$.

Proposition 2.17. *[HHR, Proposition 11.28] This pair $(MU^{((G))}, F)$ is universal in the sense that*

$$Hom_{G,\,gr}\left(\pi_*^u\left(MU^{((G))}\right), R_*\right) \simeq \left\{ \begin{array}{c} \textit{Formal groups over } R_* \textit{ with a } G\textit{-action} \\ \textit{extending the conjugation action by } C_2 \subseteq G \end{array} \right\}$$

References

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