1. Talk 1 Part 1: History of the Kervaire Invariant

This is the first talk about Hill–Hopkins–Ravenel's solution to the Kervaire invariant problem. It's divided into two parts. For the first part, I will talk about some history of the Kervaire invariant one problem. For the second part, I will give an outline of Hill–Hopkins–Ravenel's proof.

1.1. The Kervaire invariant.

In 2009, Hill–Hopkins–Ravenel proved the following theorem:

Theorem 1.1 (Hill–Hopkins–Ravenel)**.** *If M is a stably framed, smooth, closed manifold of Kervaire invariant one, then the dimension of M is 2, 6, 14, 30, 62, or 126.*

In order to make sense of this theorem, we need to first define the Kervaire invariant, which is an invariant of framed manifolds.

Recall that if M is a manifold, then I can imagine M is embedded in some large Euclidean space. A framing for M is an isomorphism of its stable normal bundle with the trivial bundle. Furthermore, if $\dim M = 4k+2$, then Kervaire used the framing to construct a function

$$\phi : H^{2k+1}(M; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2,$$

$$\phi(x+y) = \phi(x) + \phi(y) + \langle x, y \rangle.$$

Here, $\langle x, y \rangle$ is the intersection number of x and y.

The function ϕ is quadratic. In particular, this implies that $|\phi^{-1}(0)| \neq |\phi^{-1}(1)|$. The Arf invariant of ϕ is defined as follows:

$$Arf(\phi) = \begin{cases} 0 & \text{if } |\phi^{-1}(0)| > |\phi^{-1}(1)|, \\ 1 & \text{if } |\phi^{-1}(0)| < |\phi^{-1}(1)|. \end{cases}$$

Some people also call this the democratic invariant.

Definition 1.2. The *Kervaire invariant* of a framed manifold M of dimension $(4k+2)$ is the Arf invariant of the quadratic function ϕ:

$$\Phi(M) := Arf(\phi).$$

The Kervaire invariant is a framed cobordism invariant. In fact, it is one of the most fundamental invariants in algebraic and differential topology.

Question 1.3. In which dimensions is there a framed manifold with Kervaire invariant one?

Theorem 1.1 shows that the dimension can only be 2, 6, 14, 30, 62, and possibly 126.

The solution of the Kervaire invariant problem has a significant impact in both differential topology and homotopy theory. I would like to spend some time to talk about some history and explain why it is significant. More specifically, I want to explain the critical role it plays in the classification of smooth structures in high dimensional topology.

Definition 1.4. A **homotopy n-sphere** is a closed manifold that is homotopy equivalent to S^n.

Note that by Stephen Smale's proof of the generalized Poincaré conjecture for $n \geq 5$, every homotopy n-sphere is homeomorphic to S^n for $n \geq 5$. The question is, are they all diffeomorphic to S^n, with its usual smooth structure? Are there any exotic smooth structures on S^n?

In 1956, Milnor found a homotopy 7-sphere which is not diffeomorphic to the sphere S^7, with its usual smooth structure. Seven years later, in 1963, Kervaire and Milnor, using the theory of h-cobordism, actually calculated the number of homotopy n-spheres in terms of the homotopy groups of spheres, modulo the Kervaire invariant problem. This is the story I want to explain. From the origin of differential topology, you already see a connection between classifying manifolds and homotopy theory.

1.2. **Framed manifolds and homotopy theory.** Given a map $f : S^{n+k} \to S^n$ and suppose $P \in S^n$ is a point such that f is transverse to P. Consider

$$M^k := f^{-1}(P) \subset S^{n+k}.$$

It turns out that M^k has a natural framing, which makes it a k-dimensional framed submanifold of S^{n+k}.

This construction is the Pontryagin–Thom construction, and it sets up an isomorphism

$$\Omega^{fr}_{k,n} \cong \pi_{n+k}(S^n),$$

where $\Omega^{fr}_{k,n}$ denotes the framed cobordism classes of k-dimensional closed framed submanifolds of S^{n+k}.

If n is large enough ($n > k+1$), then both groups are independent of n. The isomorphism above becomes an abelian group isomorphism

$$\Omega^{fr}_k \cong \pi^{st}_k S^0$$

between the cobordism classes of stably framed k-manifolds (group action is disjoint union) to the kth stable homotopy groups of spheres.

This work, which was done by Pontryagin in the 1930s, establishes a very deep relationship between homotopy theory and geometry.

Question 1.5. In which dimensions is every framed manifold cobordant to a homotopy sphere?

Theorem 1.1 shows that except in dimensions 2, 6, 14, 30, 62, and possibly 126, every framed manifold is framed cobordant to a homotopy sphere.

1.3. **Kervaire–Milnor: Groups of homotopy spheres.** Denote Θ_n to be the group of homotopy n-spheres up to diffeomorphism, with the group action taking connected sums. By Stephen Smale's proof on the the generalized Poincaeé conjecture, every homotopy n-sphere is homeomorphic to the n-sphere for $n \geq 5$.

If Σ^n is a homotopy n-sphere, then it can be framed. A framing F gives an element

$$[\Sigma^n, F] \in \Omega^{fr}_n = \pi^{st}_n.$$

Furthermore, suppose F_1 and F_2 are two framings of Σ^n, then their difference is

$$[\Sigma^n, F_1] - [\Sigma^n, F_2] = [S^n, \phi]$$

for some framing ϕ of S^n.

Set $J_n \subset \pi_n^{st}$ to be the subgroup consisting of $[S^n, \phi]$, where ϕ is a framing on S^n. This is the image of the J-homomorphism

$$\pi_n O \longrightarrow \pi_n^{st} S^0.$$

The discussion above shows that our construction defines an element

$$P(\Sigma^n) = [\Sigma^n, F] \in \pi_n^{st} S^0 / J_n =: \text{coker } J_n.$$

This produces a well-defined homomorphism

$$P : \Theta_n \longrightarrow \text{coker } J_n.$$

Given this homomorphism, it is natural to ask what is its kernel and cokernel.

It turns out that there is an exact sequence

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \text{coker } J_n,$$

where Θ_n^{bp} is the subgroup of Θ_n consisting of manifolds which bound a stably parallelizable $(n+1)$-manifold.

Kervaire and Milnor showed that the map

$$\Theta_n \longrightarrow \text{coker } J_n$$

is surjective unless (possibly) $n = 4k+2$, in which case there is an exact sequence

$$\Theta_n \longrightarrow \text{coker } J_n \xrightarrow{\Phi} \mathbb{Z}/2.$$

Here, the rightmost map is the Kervaire invariant. Note that surjectivity here is equivalent to the assertion that every stably framed n-manifold is framed cobordant to a homotopy sphere (Question 1.5).

Furthermore, Kervaire and Milnor were also able to determine the group Θ_n^{bp}. When n is even, this group is trivial. Θ_{4k-1}^{bp} is a cyclic group whose order has been completely determined in terms of the Bernoulli numbers:

$$|\Theta_{4k-1}^{bp}| = a_k 2^{2k-2}(2^{2k-1} - 1) \text{ numerator}(B_k/4k).$$

In the expression above, B_k is the kth Bernoulli number, and a_k is 1 if k is even and 2 if k is odd. Building on work of Adams, they produced a formula for $|\Theta_{4k-1}|$.

They were unable to determine the group Θ_{4k+1}^{bp}, however. They could only show that there is an exact sequence

$$0 \longrightarrow \Theta_{4k+2} \longrightarrow \text{coker } J_{4k+2} \xrightarrow{\Phi} \mathbb{Z}/2 \longrightarrow \Theta_{4k+1}^{bp} \longrightarrow \Theta_{4k+1} \longrightarrow 0.$$

The solution of the Kervaire invariant problem solves this exact sequence and completes the puzzle. Theorem 1.1 shows that $\Theta_{4k+1}^{bp} = \mathbb{Z}/2$ except when $4k+2$ is one of 2, 6, 14, 30, 62, or possibly 126, in which case it becomes trivial. Roughly speaking, this is saying that the groups Θ_{4k+1} and Θ_{4k+2} are twice as large as they might have been.

1.4. **A problem in homotopy theory.** At the time that the Kervaire–Milnor paper was written, the status of the Kervaire invariant problem was far from certain. There were known to be framed manifolds of Kervaire invariant one in dimensions 2, 6, and 14. There were known to be no framed manifolds of Kervaire invariant one in dimensions 10 and 18. That was it.

Furthermore, in the early 1960s, the relationship of Kervaire's invariant to the homotopy groups of spheres was unclear. The next piece of the puzzle was unlocked by Browder using homotopy theory.

Theorem 1.6 (Browder 1969). *The Kervaire invariant of a framed n-manifold is zero unless n is of the form $2^{k+1}-2$, and in that case there is a framed manifold of Kervaire invariant one if and only if there is an element $\theta_j \in \pi_{2^{j+1}-2}S^0$ represented at the E_2-term of the classical Adams spectral sequence by the class h_j^2.*

For $j > 0$, the element h_j represents a potential element in $\pi^{st}_{2^j-1}S^0$ of Hopf invariant one. Only h_j, $j \leq 3$ survive the Adams spectral sequence. The work of Barratt, Jones, Mahowald, and Tangora showed that θ_j exists for $j \leq 5$.

To this end, the theorem that Hill–Hopkins–Ravenel proved is the following:

Theorem 1.7 (Hill–Hopkins–Ravenel). *For $j \geq 7$, the class $h_j^2 \in Ext_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ does not represent an element of the stable homotopy groups of spheres. In other words, the Kervaire invariant elements θ_j do not exist for $j \geq 7$.*

Remark 1.8. As we will see in the next section, their method of proof only show that h_j^2 does not survive to the E_∞-page for $j \geq 7$, but sheds no light on the length of the differential that these classes support.

2. Talk 1 Part 2: Outline of HHR's Proof

In light of Browder's reduction of the Kervaire invariant problem to homotopy theory, the statement that Hill–Hopkins–Ravenel proved is the following:

Theorem 2.1 (Hill–Hopkins–Ravenel). *For $j \geq 7$, the class $h_j^2 \in Ext_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ does not represent an element of the stable homotopy groups of spheres. In other words, the Kervaire invariant elements θ_j do not exist for $j \geq 7$.*

Their proof builds on the strategy used by Ravenel in his 1978 solution of the odd primary Kervaire invariant problem and it marshals three major developments in homotopy theory:

(1) Chromatic homotopy theory;
(2) The theory of structured ring spectra;
(3) Equivariant homotopy theory.

More specifically, they used equivariant homotopy theory to construct a spectrum Ω, which they called the **detecting spectrum**. They subsequently proved the following results:

Theorem 2.2 (The Detection Theorem). *If θ_j exists, then it has nonzero image in $\pi_{2^{j+1}-2}\Omega$ under the Hurewicz map $\pi_*S^0 \longrightarrow \pi_*\Omega$.*

Theorem 2.3 (The Periodicity Theorem). *The homotopy groups $\pi_*\Omega$ is periodic, with period 256: $\pi_*\Omega \cong \pi_{*+256}\Omega$.*

Theorem 2.4 (The Gap Theorem). *The group $\pi_{-2}\Omega$ is zero.*

These three theorems immediately imply Theorem 2.1. The argument is as follows: by the Detection Theorem, if θ_j^2 exists, then it has a non-zero Hurewicz image in $\pi_{2^{j+1}-2}\Omega$. However, the Periodicity Theorem and the Gap Theorem implies that $\pi_i\Omega = 0$ for all $i \equiv -2 \pmod{256}$. Therefore, θ_j does not exist for all $j \geq 7$.

2.1. **What is Ω?** What is the detecting spectrum Ω? How did Hill, Hopkins, and Ravenel construct it?

The starting point of their construction is $MU_{\mathbb{R}}$, the Real bordism spectrum of Landweber, Fujii, and Araki. The spectrum $MU_{\mathbb{R}}$ is a C_2-equivariant spectrum. Its underlying spectrum MU and the C_2-action on MU is coming from complex conjugation.

Classically, $\pi_* MU = \mathbb{Z}[x_i \mid i \geq 1]$, where

$$x_i : S^{2i} \longrightarrow MU.$$

It turns out that each of these x_i generators can be refined to become C_2-equivariant maps

$$\bar{x}_i : S^{i\rho_2} \longrightarrow MU_{\mathbb{R}}.$$

The spectrum that is of interest to us is $MU^{((C_8))} := N_{C_2}^{C_8} MU_{\mathbb{R}}$. It is a C_8-equivariant spectrum. It's underlying spectrum is $MU \wedge MU \wedge MU \wedge MU$, with the C_8-action sending

$$(a, b, c, d) \longmapsto (\bar{d}, a, b, c).$$

The underlying homotopy groups of $MU^{((C_8))}$ has the following form:

$$\pi_*^u MU^{((C_8))} = \mathbb{Z}[C_8 \cdot r_i \mid i \geq 1].$$

Here, $C_8 \cdot r_i$ denotes the set

$$\{r_i, \gamma r_i, \gamma^2 r_i, \gamma^3 r_i\},$$

with $\gamma^8 = 1$. In other words, there are four generators in each even degree, and the C_8-action is permuting them by sending one to the next.

To get somewhere, we need to invert an element $D \in \pi_{19\rho_8}^{C_8} MU^{((C_8))}$ and form the spectrum

$$\widetilde{\Omega} := D^{-1} MU^{((C_8))}.$$

This is a C_8-equivariant spectrum. The detecting spectrum Ω is defined to be the C_8-homotopy fixed point spectrum of $\widetilde{\Omega}$:

$$\Omega := \widetilde{\Omega}^{hC_8}.$$

Inverting the element D is necessary in order to make the resulting homotopy fixed point spectrum periodic. This is something that Mingcong will talk about.

The Detection Theorem involves computation with the classical Adams Novikov spectral sequence. It builds on Ravenel's 1978 proof of the odd primary Kervaire invariant. Guchuan will talk about this.

The Periodicity Theorem and the Gap Theorem involve genuine equivariant homotopy theory. Both proofs use the *slice filtration*, which is a novel equivariant refinement of the Postnikov tower. This is the topic of the next talk. The equivariant slice filtration is analogous to the slice filtration in motivic homotopy theory and it generalizes the filtration described by Dugger regarding Atiyah's Real K-theory $K\mathbb{R}$.

As a remark, the Detection Theorem and the Periodicity Theorem are proven for the homotopy fixed point spectrum $\widetilde{\Omega}^{hC_8}$. The Gap Theorem, however, is proven for the fixed points spectrum $\widetilde{\Omega}^{C_8}$. To tie these results together, HHR exploits the relatively simple fact that the map

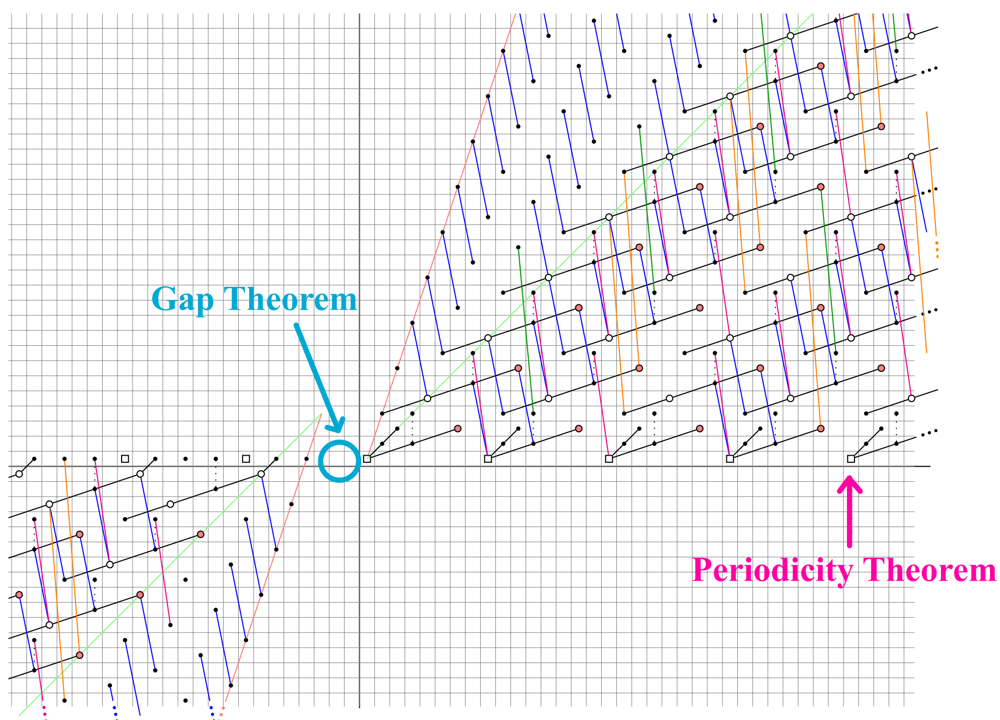
$$\widetilde{\Omega}^{C_8} \longrightarrow \widetilde{\Omega}^{hC_8}$$

is a homotopy equivalence. This is called the *Homotopy Fixed Point Theorem*. In fact, D, the generator we invert, is chosen to make sure that we also have this equivalence.

Here is the plan for the next five talks:

(1) For the next talk, I am going to set up the slice filtration. Then we are going to compute the slice tower for $\widetilde{\Omega}$. This is a relatively technical part of their paper, called the *Slice Theorem* and the *Reduction Theorem*. HHR goes a long way to prove this. The upshot is that these two theorems show that the slices for $\widetilde{\Omega}$ are nice. More specifically, each slice is a wedge of regular suspensions of $H\underline{\mathbb{Z}}$, the constant Mackey functor.

(2) Once we have that, Mingcong is going to compute their homotopy groups. They can be completely computed, and doing so will immediately prove the Gap Theorem. Moreover, we will get the E_2-page of the slice spectral sequence. Analyzing a small portion of the slice spectral sequence will prove the Periodicity Theorem.



(3) Finally, to wrap things up, Guchuan will talk about the Detection Theorem, which involves computation with the Adams–Novikov spectral sequence and ties the picture up with chromatic homotopy.

3. Talk 2 Part 1: The Slice Filtration

The slice tower is an equivariant analogue of the Postnikov tower.

3.1. **The classical Postnikov tower.** Let X be a space or a spectrum. Recall that the Postnikov tower for X is the following:

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & \nearrow & P^n X & \longleftarrow & \Sigma^n H(\pi_n X) \\
 & & \downarrow & & \\
X & \longrightarrow & P^{n-1} X & \longleftarrow & \Sigma^{n-1} H(\pi_{n-1} X) \\
 & & \downarrow & & \\
 & & \vdots & &
\end{array}$$

There are maps $X \to P^n X$ for every n. The limit and the colimit of the tower are as follows:

$$\begin{aligned} \varprojlim P^n X &\simeq X \\ \varinjlim P^n X &\simeq * \end{aligned}$$

The classical Postnikov tower has the property that $P^n X$ has no higher homotopy groups above dimension n. That is to say,

$$\pi_k P^n X = 0$$

for all $k > n$. One can therefore view the maps $P^n X \to P^{n-1} X$ as killing off higher homotopy groups.

In fact, $P^n X$ is the universal spectrum such that

$$\begin{aligned} \mathrm{Map}(S^{>n}, P^n X) &\simeq * \\ \mathrm{Map}(S^k, P^n X) &= \mathrm{Map}(S^k, X), \;\; k \le n \end{aligned}$$

I want to dwell on the first condition a little bit more. Notice that if Y is any spectrum that is obtained from $S^{>n}$ via taking cofibers, colimits, or extensions, then

$$\mathrm{Map}(Y, P^n X) \simeq *.$$

This can be rephrased as follows: let $\tau_{\ge n+1}$ denote the full subcategory obtained from the spheres $\{S^k \mid k > n\}$ by closing up under extensions, cofibers, and colimits. If $Y \in \tau_{\ge n+1}$, then

$$\mathrm{Map}(Y, P^n X) \simeq *.$$

As a warning, the category $\tau_{\ge n+1}$ is not closed under taking fibers and limits. In particular, we can't take desuspensions. For example, while it is true that $\mathrm{Map}(S^{n+1}, P^n X) \simeq *$, it is *not* true that $\mathrm{Map}(S^n, P^n X) \simeq *$.

The category $\tau_{\ge n+1}$ is known by another name. It is the full subcategory of n-connected spectra. $P^n X$ is the Dror nullification with respect to $\tau_{\ge n+1}$.

3.2. **The slice tower.** In light of the classical Postnikov tower, the construction of the slice tower will follow the same logic. In the equivariant context, the general rubric for constructing a tower for a G-spectrum X is as follows:

(1) Replace classical contractibility of mapping spaces to equivariant contractibility: a G-space is equivariantly contractible if it is H-equivariant contractible for all $H \subseteq G$.
(2) Replace $\tau_{\geq n+1} = \{S^k \,|\, k \geq n+1\}$ by a family of equivariant spheres $\{S^V\}$.
(3) Apply Dror nullification functor with respect to $\tau_{\geq n+1}$ to obtain a tower.

There are some freedom to step (2). We can choose any collection of spheres we want. As an example, we can choose the spheres

$$G/H_+ \wedge S^k \simeq G_+ \wedge_H S^k$$

for $k \geq n+1$, and let $\tau_{\geq n+1}$ be the full subcategory containing these spheres that is closed under taking cofibers, colimits, and extensions. The category $\tau_{\geq n+1}$ is the subcategory of equivariantly n-connected spectra. After applying Dror nullification, the tower associated to this choice of spheres is the equivariant Postnikov tower:

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & P^n X & \longleftarrow & \Sigma^n H(\underline{\pi}_n X) \\
 & \nearrow & \downarrow & & \\
X & \longrightarrow & P^{n-1} X & \longleftarrow & \Sigma^{n-1} H(\underline{\pi}_{n-1} X) \\
 & & \downarrow & & \\
 & & \vdots & &
\end{array}$$

Now we are finally ready to define the slice tower. The collection of spheres that we choose is going to contain spheres of the form $G_+ \wedge_H S^k$ (this is not from the definition, we will prove it later), but we will also add induced representation spheres.

Let H be a subgroup of G and k an integer. Define

$$\begin{aligned}
W(k, H) &:= G_+ \wedge_H S^{k\rho_H}, \\
\Sigma^{-1} W(k, H) &:= G_+ \wedge_H S^{k\rho_H - 1}.
\end{aligned}$$

Definition 3.1. A *slice cell* is a spectrum that is of the from $W(k, H)$ or $\Sigma^{-1} W(k, H)$ for some integer k and $H \subseteq G$.

Definition 3.2. The *dimension* of the slice cell $\Sigma^\epsilon W(k, H)$ ($\epsilon = 0, -1$) is the dimension of its underlying complex:

$$\dim(\Sigma^\epsilon W(k, H)) = k|H| + \epsilon.$$

Let $\tau_{\geq n}$ be the subcategory containing all slice cells of dimension $\geq n$ that is closed under taking cofibers, colimits, and extensions. Note that

$$\tau_{\geq n+1} \subseteq \tau_{\geq n}.$$

After applying Dror nullification to X with respect to the subcategories $\tau_{\geq n}$, we obtain a tower

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ & \nearrow & P^n X \\ & & \downarrow \\ X & \longrightarrow & P^{n-1}X \\ & & \downarrow \\ & & \vdots \end{array}$$

This is the slice tower.

The C_2-equivariant slice tower is defined by Dugger in his PhD thesis, where he analyzed Atiyah's Real K-theory $K\mathbb{R}$. The slice tower is also motivated by the motivic slice story of Voevodsky and Hopkins–Morel.

As a warning, in the motivic slice story, the subcategories $\tau_{\geq n}$ are triangulated categories that are closed under taking desuspensions. This is not the case here. If we do close $\tau_{\geq n}$ under taking desuspensions, then the slice tower becomes contractible at each stage.

3.3. **Properties of the slice tower.** It is immediate from the definition of the slice cells that they behave well under the restriction map and the induction map. More specifically, the following fact follows immediately from the definition.

Fact 3.3. Let $H \subseteq G$ be a subgroup. If $\widehat{S}$ is a G-slice cell of dimension d, then $i_H^* \widetilde{S}$ is a wedge of H-slice cells of dimension d. If $\widehat{S}$ is an H-slice cell of dimension d, then $G_+ \wedge_H \widehat{S}$ is a G-slice cell of dimension d.

Proposition 3.4. *The G-cells $G_+ \wedge_H S^n \in \tau_{\geq n}$.*

Proof. Since slice cells behave well with respect to the induction functor $G_+ \wedge_H (-)$, it suffices to show by induction on H that $G_+ \wedge_G S^n = S^n$ is in $\tau_{\geq n}$. Consider the cofiber sequence

$$S^n \longrightarrow S^{n\rho_G} \longrightarrow S^{n\rho_G}/S^n.$$

The cofiber, $S^{n\rho_G}/S^n$, is built out of induced cells of dimension $\geq n+1$. That is to say, the G-CW decomposition for $S^{n\rho_G}/S^n$ only contains G-cells of the from $G_+ \wedge_H S^k$, where $H \subsetneq G$ and $k \geq n+1$.

We can rewrite the cofiber sequence above as

$$\Sigma^{-1} S^{n\rho_G}/S^n \longrightarrow S^n \longrightarrow S^{n\rho_G}.$$

Here, $\Sigma^{-1} S^{n\rho_G}/S^n$ is built out of induced G-cells of dimension $\geq n$. By the induction hypothesis, all of these G-cells are in $\tau_{\geq n}$. Since $S^{n\rho_G} \in \tau_{\geq n}$ and $\tau_{\geq n}$ is closed under taking extensions, $S^n \in \tau_{\geq n}$. □

The proposition above shows that the subcategory of n-connected equivariant spectra is contained in $\tau_{\geq n+1}$. In other words, when we nullify X with respect to $\tau_{\geq n+1}$,

$$\underline{\pi}_k P^n X = \underline{0}$$

for all $k \geq n+1$. They are quite connected. As a consequence, the colimit of the slice tower is contractible:

$$\varinjlim P^n X \simeq *.$$

To identify the limit of the slice tower, we need the following proposition.

Proposition 3.5. *The slice cell $G_+ \wedge_H S^{m\rho_H}$ is $(m-1)$-connected.*

Proof. In the G-CW decomposition for $G_+ \wedge_H S^{m\rho_H}$, the bottom cell is $G_+ \wedge_H S^m$, which is in dimension m. All the other cells are above dimension m. The claim follows. ☐

One can use Proposition 3.5 to show that all the slice cell generators of $\tau_{\geq n}$ are $\left(\lfloor \frac{n}{|G|} \rfloor - 1\right)$-connected. Therefore, any element $Y \in \tau_{\geq n}$ is also $\left(\lfloor \frac{n}{|G|} \rfloor - 1\right)$-connected.

When we nullify X with respect to $\tau_{\geq n}$, the map

$$X \longrightarrow P^{n-1} X$$

is $\left(\lfloor \frac{n}{|G|} \rfloor - 1\right)$-connected for every n. As n-increases, the connectivity of the map increases, from which it follows that

$$\varprojlim P^n X \simeq X.$$

Let's look at the slice tower again:

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & P^n X & \longleftarrow & P^n_n X \\
 & \nearrow & \downarrow & & \\
X & \longrightarrow & P^{n-1} X & \longleftarrow & P^{n-1}_{n-1} X \\
 & & \downarrow & & \\
 & & \vdots & &
\end{array}$$

Each of the layers (the slices) $P^n_n X$ has a finite range of dimensions where it has nonzero homotopy groups. When we compute the slice spectral sequence, these slices are going to interact with each other.

Compare this with the Postnikov tower. Each of the layers also has a finite range of dimensions where it has nonzero homotopy groups. In fact, the homotopy groups are all concentrated in one single dimension. This is bad for computations because in order to write down the E_1-term, we would need to already have computed the homotopy groups of X.

The slice tower leads to the slice spectral sequence

$$E_1 = \pi^G_{t-s} P^t_t X \Longrightarrow \pi^G_{t-s} X.$$

Since the slice filtration is an equivariant filtration, we can take Mackey functor valued homotopy groups as well:

$$E_1 = \underline{\pi}_{t-s} P^t_t X \Longrightarrow \underline{\pi}_{t-s} X.$$

The first spectral sequence is a spectral sequence of Abelian groups, whereas the second spectral sequence is a spectral sequence of Mackey functors. The reader might find it more intuitive to think of it as a Mackey functor of spectral sequences: for each subgroup $H \subseteq G$, there is a spectral sequence

$$E_1 = \pi^H_{t-s} P^t_t X \Longrightarrow \pi^H_{t-s} X.$$

Between subgroups, there are restriction and transfer maps between them.

We can also use the slice tower to compute the $RO(G)$-graded homotopy groups of X:

$$E_1 = \pi^G_\bigstar P^t_t X \Longrightarrow \pi^G_\bigstar X.$$

If we are really ambitious, we can compute the Macky functor $RO(G)$-graded homotopy groups of X:

$$E_1 = \underline{\pi}_\bigstar P^t_t X \Longrightarrow \underline{\pi}_\bigstar X.$$

3.4. **Some more properties of the slices.**

Example 3.6. Consider the subcategory $\tau_{\geq 0}$, which is generated by slice cells of the form

$$\{\Sigma^\epsilon W(k, H) \mid k \cdot |H| + \epsilon \geq 0\}.$$

The slice connectivity result that we proved earlier implies that all the elements in $\tau_{\geq 0}$ are (-1)-connected. This category also contains the cells

$$W(0, H) = G_+ \wedge_H S^0 = G/H_+,$$

which generated the full subcategory of (-1)-connected spectra. It follows that $\tau_{\geq 0}$ is the full subcategory of (-1)-connected spectra.

Example 3.7. We can do the same thing for $\tau_{\geq -1}$, from which we learn that $\tau_{\geq -1}$ is the full subcategory of (-2)-connected spectra.

The two examples above imply that

$$P^{-1}_{-1} X = \Sigma^{-1} H\underline{\pi}_{-1} X.$$

This is in the sense that the (-1)-slice is giving us the same information as the (-1)-Postnikov layer. The good news is that this is the only dimension where this happens.

It turns out that

$$P^0_0 S^0 = H\underline{\mathbb{Z}}.$$

This is where we see that the slices are behaving better than the Postnikov sections. The 0th Postnikov section for S^0 is $H\underline{A}$, where A is the Burnside Mackey functor. What happened here is that we need slice cells of the form $\Sigma^{-1} W(k, H)$. By killing them off, we essentially quotiented out $H\underline{A}$ and obtained a much easier spectrum, which is $H\underline{\mathbb{Z}}$.

As an another observation, note that smashing with $S^{k\rho_G}$ sends a slice cell to another slice cell of dimension $k|G|$ higher:

$$S^{k\rho_G} \wedge (G_+ \wedge_H S^{\ell\rho_H}) = G_+ \wedge_H (S^{(k\cdot|G/H|+\ell)\rho_H}).$$

This establishes a bijection between the slice cells of the corresponding dimensions and produces a map

$$\Sigma^{k\rho_G} : \tau_{\geq n} \longrightarrow \tau_{\geq n+k\cdot|G|}.$$

It follows from this that we have an equivalence

$$P^{n+k\cdot|G|}\Sigma^{k\rho_G} X = \Sigma^{k\rho_G} P^n X.$$

4. Talk 2 part 2: the Slice Theorem

We are now finally ready to describe the slices of $MU^{((G))}$. From now on, we will set $G = C_{2^n}$, the cyclic group of order 2^n. Let γ be the generator of G. The underlying spectrum of $MU^{((G))}$ is a smash product of 2^{n-1}-copies of MU. There exist good generators, defined in terms of formal group laws, such that

$$\pi_*^u MU^{((G))} = \mathbb{Z}[G \cdot r_1, G \cdot r_2, \ldots].$$

Here, the degree of r_i is $2i$, the notation $G \cdot r_i$ represents the set

$$\{r_i, \gamma r_i, \ldots, \gamma^{2^{n-1}-1} r_i\}.$$

The generator γ of G acts on r_i as follows:

$$\gamma(\gamma^j r_i) = \begin{cases} \gamma^{j+1} r_i & j < 2^{n-1} - 1, \\ (-1)^i r_i & i = 2^{n-1} - 1. \end{cases}$$

What this means is that non-equivariantly, we can write the Postnikov associated graded as

$$H\mathbb{Z} \wedge \left(\bigvee_p S^{|p|} \right),$$

where p ranges over all monomials in the polynomial ring $\mathbb{Z}[G \cdot r_1, G \cdot r_2, \ldots]$.

The action of G on $MU^{((G))}$ induces an action of G on $\pi_*^u MU^{((G))}$. In particular, it acts on the monomials p. Equivariantly, we might try to group the spheres $S^{|p|}$ together based on how the group is permuting the monomial p. If we do this, we get

$$H\mathbb{Z} \wedge \left(\bigvee_{\text{orbits of monomials}} \left(\bigvee_{G/\text{stab}(p)} S^{|p|} \right) \right).$$

Here, by the stabilizer of a monomial p, we really mean the stabilizer of a monomial modulo 2. This is because $\gamma^{2^{n-1}} p = -p$, but the map corresponding to the monomials p and $-p$ are really carried by the same sphere. So if we are thinking of monic things, we can't really distinguish p and $-p$.

What makes the whole thing work is that each of the r_i-generators, which represents a non-equivariant map

$$r_i : S^{2i} \longrightarrow MU^{((G))},$$

can be refined to become C_2-equivariant maps

$$\bar{r}_i : S^{i\rho_2} \longrightarrow MU^{((G))}.$$

This will ultimately let us replace $S^{|p|}$ by

$$S^{\frac{|p|}{|\text{stab}(p)|}\rho_{\text{stab}(p)}}$$

and $\bigvee_{G/\text{stab}(p)} S^{|p|}$ will be replaced by

$$G_+ \wedge_{\text{stab}(p)} S^{\frac{|p|}{|\text{stab}(p)|}\rho_{\text{stab}(p)}}.$$

This is an induced slice cell!

I won't spell out the technical details, but the upshot of all this is that the slice filtration looks very similar to the non-equivariant Postnikov filtration.

Theorem 4.1 (The Slice Theorem)**.** *The slice associated graded of $MU^{((G))}$ is*

$$H\underline{\mathbb{Z}} \wedge \left(\bigvee_{orbits\ of\ monomials} G_+ \wedge_{stab(p)} S^{\frac{|p|}{|stab(p)|}\rho_{stab(p)}} \right),$$

where the oribits of monomials ranges through orbits of monomials in $\pi_*^u MU^{((G))} \otimes \mathbb{Z}/2$.

It follows from the slice theorem that the slice associated graded for $\Sigma^{-k\rho_G} MU^{((G))}$ has the same form.

Note that $C_2 \subseteq \text{stab}(p) \subset C_{2^n}$. This is because C_2 sends p to $-p$, and we are working modulo 2. As a consequence of this, there are no slice cells of the form

$$G_+ \wedge S^n.$$

Moreover, we see that the slices of $MU^{((G))}$ are of the form

$$H\underline{\mathbb{Z}} \wedge (\text{regular slice cells}).$$

The slice cells of the form $\Sigma^\epsilon W(k, H)$.

Mingcong is going to tell us how to compute the homotopy groups of the wedge summands and show that various homotopy groups vanish. This will prove the gap theorem and produce the E_1-page of the slice spectral sequence.