# THE SLICE DIFFERENTIAL THEOREM, THE PERIODICITY THEOREM AND THE HOMOTOPY FIXED POINT THEOREM

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## 1. Preliminary computation

From [HHR, Thm 6.1] and the proof of [HHR, Lemma 6.7] we see that the slices of $MU^{((C_{2^n}))}$ has the form

$$H\underline{\mathbb{Z}}[C_{2^n} \cdot \overline{r}_1, C_{2^n} \cdot \overline{r}_2, ...].$$

The definition of the notation is in [HHR, Section 2.4], but we would like to give a few example of what does it mean.

**Example 1.1.** *For $G = C_2$, the slices of $MU_{\mathbb{R}}$ are $H\underline{\mathbb{Z}}[\overline{r}_1, \overline{r}_2, ...]$. The 0 slice is $H\underline{\mathbb{Z}}$, the 2 slice is $\Sigma^{\rho_2} H\underline{\mathbb{Z}}$, generated by $\overline{r}_1$ and the 4 slice is two copies of $\Sigma^{2\rho_2} H\underline{\mathbb{Z}}$, generated by $\overline{r}_1^2$ and $\overline{r}_2$.*

*For $G = C_4$, the slices of $MU^{((C_4))}$ are $H\underline{\mathbb{Z}}[C_4 \cdot \overline{r}_1, C_4 \cdot \overline{r}_2, ...]$. The 0 slice is $H\underline{\mathbb{Z}}$, the 2 slice is $C_4/C_{2+} \wedge \Sigma^{\rho_2} H\underline{\mathbb{Z}}$, generated by $\overline{r}_1$ and the 4 slice is the wedge of $\Sigma^{\rho_4} H\underline{\mathbb{Z}}$ and two copies of $C_4/C_{2+} \wedge \Sigma^{2\rho_2} H\underline{\mathbb{Z}}$, where the former is generated by $N(\overline{r}_1)$ and the latter by $\overline{r}_1^2$ and $\overline{r}_2$.*

With the understanding of slices, we can start to compute slice spectral sequence. In general, we want to compute the slice spectral sequence as an $RO(G)$-graded spectral sequence of Mackey functors whenever it is possible. There are two different kinds of slices

(1) Induced slice of the form $G_+ \wedge_H \Sigma^{k\rho_H} H\underline{\mathbb{Z}}$.
(2) Non-induced slices $\Sigma^{k\rho_G} H\underline{\mathbb{Z}}$.

The homotopy of induced slices can be computed in the subgroup $H$ first then induce up. We need the following definition.

**Definition 1.2.** *Let $H \subset G$ and $\underline{M}$ be a $H$-Mackey functor. The induced Mackey functor $\uparrow_H^G \underline{M}$ is the $G$-Mackey functor defined by the composition*

$$B_G \xrightarrow{i_H^*} B_H \xrightarrow{\underline{M}} Ab.$$

*where the first functor is the forget functor from $G$-sets to $H$-sets.*

**Proposition 1.3.**

$$\underline{\pi}_V(G_+ \wedge_H X) \cong \uparrow_H^G \underline{\pi}_{i_H^* V}(X).$$

This is a direct consequence of Wirthmüller isomorphism.

**Proposition 1.4.** *Let $G = C_{2^n}$, then 2-locally there are essentially $n+1$ non-isomorphic irreducible $G$-representations. They are $1$, the trivial representation, $\sigma$, the sign representation and $\lambda_k$, the $\mathbb{R}^2$-representation given by rotating $\frac{2^{k+1}\pi}{2^n}$ for $0 \leq k < n-1$.*

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We can visualize some cellular structure. The idea of constructing such a simple cellular structure is the following lemma.

**Lemma 1.5.** *Let $K \subset H \subset G$, then there is no $G$-sets map $G/H \to G/K$.*

This lemma tells us that we should build $G$-CW-structure representation spheres out of sequences of fixed points, by smaller and smaller subgroups.

**Example 1.6.** *$G = C_2$, $V = n\sigma$ for $n \geq 0$. $S^V$ has a cellular structure as following: Its $0$-cell is fixed, and it has one free cell in each dimension between $1$ and $n$. Therefore the cellular chain complex is*

$$\underline{\mathbb{Z}} \longleftarrow \underline{\mathbb{Z}[C_2]} \longleftarrow \underline{\mathbb{Z}[C_2]} \longleftarrow \dots \longleftarrow \underline{\mathbb{Z}[C_2]}$$

*one can compute its homology by the fact that the underlying chain complex compute the homology of ordinary $S^n$.*

In general, in $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})$ there are a few important elements that generates an important part of the coefficient ring.

**Definition 1.7.**
(1) *Let $a_V$ be the map $a_V : S^0 \to S^V$ by embedding $S^0$ as $\{0, \infty\}$. Its Hurewicz image in $\underline{H}_0(S^V; \underline{\mathbb{Z}})$ shares the same name.*
(2) *If $V$ is orientable (i.e. the defining map $G \to O(V)$ factors through $SO(V)$), then there is a unique class $u_V \in \underline{H}_{|V|}(S^V; \underline{\mathbb{Z}})$ restricts to the preassigned underlying generator of $\tilde{H}_{|V|}(S^V)$.*

**Theorem 1.8.** *Let $G = C_{2^n}$, then as a ring, $\pi^G_\bigstar(H\underline{\mathbb{Z}})$ for $-\bigstar$ an actual representation is the following*

$$\mathbb{Z}[a_\sigma, a_{\lambda_k}, u_{2\sigma}, u_{\lambda_k}]/(2^{n-k}a_{\lambda_k}, 2a_\sigma, \textit{gold relations})$$

*where the gold relations are $a_{\lambda_k}u_{\lambda_l} = 2^{k-l}a_{\lambda_l}u_{\lambda_k}$ for $k > l$.*

**Exercise 1.9.** *Let $G = C_4$, compute $S^\lambda, S^{2\lambda}, S^{2\sigma+2\lambda}$ and show all possible multiplications by $a$ and $u$.*

Let $V$ be an actual representation, $\tilde{H}_*(S^{-V}; \underline{\mathbb{Z}}) \cong \tilde{H}^{-*}(S^V; \underline{\mathbb{Z}})$, therefore one can compute via similar cochain complexes. Direct computation gives the following:

**Proposition 1.10.** *For $G = C_{2^n}$ and $k < 0$, $\tilde{H}_{-2}(S^{k\rho_G}; \underline{\mathbb{Z}}) = 0$.*

This is an ingredient of the gap theorem.

Before we start computing the slice spectral sequence, there is one more facts we need.

**Proposition 1.11.**
(1) *Fix $k > l$, $a_{\lambda_k}$ is the composition of $a_{\lambda_l}$ and a map $\frac{a_{\lambda_k}}{a_{\lambda_l}} : S^{\lambda_l} \to S^{\lambda_k}$. Therefore inverting $a_{\lambda_k}$ will make $a_{\lambda_l}$ invertible. Specially, inverting $a_\sigma$ makes all $a_V$ invertible.*
(2) *$\pi^G_\bigstar(a_\sigma^{-1}H\underline{\mathbb{Z}}) = \mathbb{Z}/2[a^\pm_{\lambda_k}, a^\pm_\sigma, u_{2\sigma}]$.*
(3) *$\underline{\pi}_\bigstar(a_\sigma^{-1}G/H_+ \wedge H\underline{\mathbb{Z}}) = \underline{0}$.*

**Remark 1.12.** *The geometric reason of this proposition is the fact that $S^{\infty\sigma} \wedge S^V \simeq S^{\infty\sigma}$ if $V^G = 0$.*

## 2. the Slice theorem and some permanent cycles

With all the preparation, we can start to compute some differentials in the slice spectral sequence of $N_2^{2^n}MU_{\mathbb{R}}$. In general, there are a lot of slice differentials in the slice spectral sequence, but for the purpose of the Kervaire invariants, we only need one family of them.

**Theorem 2.1** ([HHR, Thm 9.9]). *Let $G = C_{2^n}$, $g = |G|$ and $\overline{\rho}_G = \rho_G - 1$ the reduced sign representation, then $d_i(u_{2\sigma}^{2^{k-1}}) = 0$ for $i < r = 1 + (2^k - 1)g$ and*

$$d_r(u_{2\sigma}^{2^{k-1}}) = N(\overline{r}_{2^k-1})a_{\overline{\rho}_G}^{2^k-1}a_\sigma^{2^k}.$$

This session is mostly about the proof of this theorem.

**Definition 2.2.** *Let $R$ be a $G$-commutative ring spectrum and $x \in \pi_V^H(R)$. $N(x) \in \pi^G_{ind_H^G V}(R)$ the internal norm of $x$ is defined by*

$$S^{ind_H^G V} \simeq N_H^G(S^V) \longrightarrow N_H^G(i_H^* R) \longrightarrow R.$$

The geometric input we need to resolving these differentials is exactly the geometric fixed points.

**Proposition 2.3.** (1) *The $C_2$-geometric fixed point of $MU_{\mathbb{R}}$ is $MO$.*
(2) *$\Phi^G(N_{C_2}^G X) \simeq \Phi^{C_2}(X)$ if $X$ is cofibrant.*
(3) *If $G = C_{2^n}$, then $\Phi^G(X) \simeq (a_\sigma^{-1}X)^G$*

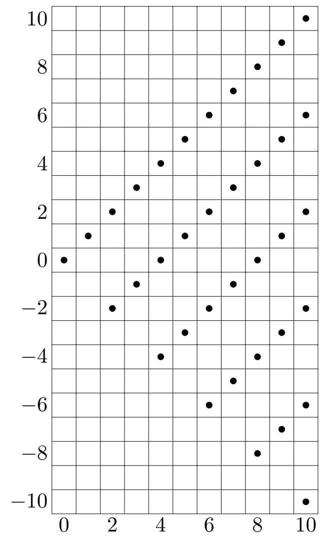
**Proposition 2.4.** *For each $G = C_{2^n}$ there is a set of elements $\overline{r}_i^G \in \pi^{C_2}_{i\rho_2}(N_2^{2^n}MU_{\mathbb{R}})$ such that*

(1) $\pi^{C_2}_{*\rho_2}(N_2^{2^n}MU_{\mathbb{R}}) = \mathbb{Z}[\overline{r}_1^G, \overline{r}_1^G, ...]$
(2) *$\Phi^G(N(\overline{r}_i^G))$ is $h_i$ if $i \neq 2^k - 1$ and $0$ if $i = 2^k - 1$. Where $h_i$ are generators of*

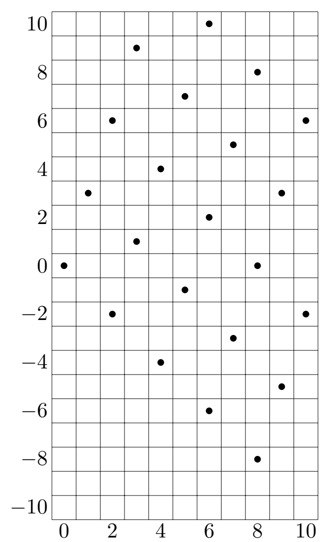
$$\pi_*(MO) = \mathbb{F}_2[h_i | i \neq 2^k - 1].$$

Now, we can invert $a_\sigma$ in each slice of $N_2^{2^n}MU_{\mathbb{R}}$ and it computes $a_\sigma^{-1}N_2^{2^n}MU_{\mathbb{R}}$, whose $H$-fixed points are trivial for $H \neq G$ (as $a_\sigma$ is null-homotopic restricting to $H$), and whose $G$-fixed point is $MO$. By the proposition above, $a_\sigma^{-1}SSS(N_2^{2^n}MU_{\mathbb{R}})$ is very simple: Induced slices contribute nothing, and non-induced slices gives $\mathbb{Z}/2[x]$ for $x$ in degree $2$, and we know this spectral sequence computes $\pi_*(MO)$.

When $G = C_2$, the $E_2$-page of the integral graded $a_\sigma^{-1}SSS(MU_{\mathbb{R}})$ is the following:



When $G = C_4$, the $E_2$-page of the integral graded $a_\sigma^{-1}SSS(MU_{\mathbb{R}})$ is the following:



Notice that these pictures come without ranks. The rank on the n-th diagonal is the number of degree $2n$ monomials in $\mathbb{Z}/2[r_1, r_2, ...]$.

From the picture, we see that when $G$ changes, the $a_\sigma$-inverted slice spectral sequence doesn't change much: their $E_2$-pages are isomorphic after reindexing.

**Proposition 2.5.** *As a ring, the integral $E_2$-page of $a_\sigma^{-1}SSS(N_2^{2^n}MU_{\mathbb{R}})$ is*

$$\mathbb{Z}/2[a_\sigma^{-2}u_{2\sigma}][N\overline{r}_1 a_{\overline{\rho}_G}^{-1}, N\overline{r}_2 a_{\overline{\rho}_G}^{-2}, ...]$$

*The $RO(G)$-graded $E_2$-page is*

$$\mathbb{Z}/2[a_{\lambda_k}^{\pm}, a_\sigma^{\pm}, u_{2\sigma}][N\overline{r}_i]$$

*for* $0 \leq k \leq n-2$ *and* $i \geq 1$.

Notice that by the choice of $\overline{r}_i$, we have $\Phi^G(N(\overline{r}_i)) = h_i$ if $i \neq 2^k - 1$ and $\Phi^G(N(\overline{r}_{2^k-1})) = 0$. This uniquely determines all the differentials in $a_\sigma$-inverted slice spectral sequence, and the differentials exactly reads as the theorem. If we write everything as modules over permanent cycles $\mathbb{Z}/2[a_\sigma^2\overline{r}_2, a_\sigma^4\overline{r}_4, ...]$, then it becomes $a_\sigma^{-1}SSS(BP_\mathbb{R})$ and it computes $\pi_*(H\mathbb{Z}_2)$.

Now that we resolve completely $a_\sigma^{-1}SSS(MU^{((C_{2^n}))})$, we can pullback these differentials along the map $SSS(MU^{((C_{2^n}))}) \to a_\sigma^{-1}SSS(MU^{((C_{2^n}))})$. The following picture shows when $G = C_4$, the $* - 2\sigma$ grading $E_2$-page of $SSS(MU^{((C_{2^n}))})$, where the class $u_{2\sigma}^2$ lives. One see immediately that in the range related to differentials on $u_{2\sigma}^2$, the spectral sequence is isomorphic to the $a_\lambda$-inverted one, at the class $a_\sigma^{-2}u_{2\sigma}$. Therefore, $u_{2\sigma}$ must support the same differential as $a_\sigma^{-2}u_{2\sigma}$. The exact same argument works for all $u_{2\sigma}^{2^k}$, thus we prove the slice differential theorem.

In the range that only contains the norm part (above the line of slope $2^n - 1$), one can think the slice spectral sequence as chopping off negative filtration elements from the $a_\sigma$-inverted one, which means that in the $a_\sigma$-inverted spectral sequence, cycles that killed by negative filtration elements become non-trivial permanent cycles. The first example of such an element is $\overline{r}_1 u_{2\sigma}^2$ in $G = C_2$: in $a_\sigma^{-1}SSS(MU_\mathbb{R})$ there is a differential $d_3(a_\sigma^{-3}u_{2\sigma}^3) = \overline{r}_1 u_{2\sigma}^2$, but such a differential cannot exist in $SSS(MU_\mathbb{R})$. Furthermore, the potential differential $d_7(\overline{r}_1 u_{2\sigma}^2) = \overline{r}_1\overline{r}_2 a_\sigma^7 = d_3(\overline{r}_2 a_{s\sigma}^4 u_{2\sigma})$ tells us that its $d_7$ target is already gone after $d_3$, and by degree reason there is no more target it can potentially hit, so we can conclude that $\overline{r}_1 u_{2\sigma}^2$ is a nontrivial permanent cycle in $SSS(MU_\mathbb{R})$.

One shall believe that the same argument works for all $N(\overline{r}_{2^k-1})u_{2\sigma}^{2^k}$ for all $G = C_{2^n}$, since in the range we consider, the spectral sequences are exactly the same after reindex.

**Corollary 2.6.** *The classes* $N(\overline{r}_{2^k-1})u_{2\sigma}^{2^k}$ *are nontrivial permanent cycles in* $MU^{((C_{2^n}))}$.

These classes give us enough material to obtain periodicity.

## 3. THE PERIODICITY THEOREM

The motivation and the first example of the periodicity theorem is the $C_2$-spectrum $k\mathbb{R} := MU_\mathbb{R}/(\overline{r}_2, \overline{r}_3, ...)$. Its associated graded slice is

$$H\underline{\mathbb{Z}}[\overline{r}_1]$$

and the slice spectral sequence is determined by $d_3(u_{2\sigma}) = a_\sigma^3\overline{r}_1$. After $d_3$ we see that by degree reason $u_{2\sigma}^2$ is a permanent cycle. Notice that $\overline{r}_1^4 u_{2\sigma}^2$ is a permanent cycle in $\pi_8^{C_2}(k\mathbb{R})$. Let $K\mathbb{R} := \overline{r}_1^{-1}k\mathbb{R}$, then the map

$$\Sigma^8 K\mathbb{R} \xrightarrow{\overline{r}_1^4 u_{2\sigma}^2} K\mathbb{R}$$

is an underlying equivalence, since $u_{2\sigma}$ restricts to $1$ and $\overline{r}_1$ is invertible. Therefore, it induces a weak equivalence

$$\Sigma^8 KO \longrightarrow KO\ ,$$

which is the classical real Bott periodicity.

The second example is $MU_\mathbb{R}$. By the last corollary, we see that $\overline{r}_1 u_{2\sigma}^2$ is a permanent cycle in $MU_\mathbb{R}$. That means, even though $u_{2\sigma}$ supports a differential in $SSS(MU_\mathbb{R})$, it becomes a permanent cycle in $\overline{r}_1^{-1}MU_\mathbb{R}$. By the same argument, we see that the homotopy fixed point $(\overline{r}_1^{-1}MU_\mathbb{R})^{hC_2}$ is 8-periodic.

So, how do we play a similar game to $MU^{((C_{2^n}))}$? We wish to make a power of a certain $u$ a permanent cycle, and then use $N(\bar{r}_1)$-power to make it into the integer degrees. That means, this $u$ must be a power of $u_{2\rho_G}$. To make a power of $u_{2\rho_G}$ a permanent cycle, we first need the following decomposition.

**Proposition 3.1.**
$$u_{2\rho_G} = \prod_{0\neq H\subset G} N_H^G(u_{2\sigma_H}^{h/2}).$$
*Where $\sigma_H$ is the sign representation of the subgroup $H \subset G$.*

This means that if we can make some power of $u_{2\sigma_H}$ a permanent cycle for each $0 \neq H \subset G$, then we can make a power of $u_{2\rho_G}$ a permanent cycle. By the same argument in $MU_{\mathbb{R}}$, we only need to invert $N(\bar{r}_{2^i-1}^H)$ for each $H$ and some $i$.

**Example 3.2.**
(1) *Let $G = C_4$ and we invert $D = N(\bar{r}_1^{C_2})N(\bar{r}_1^{C_4})$. That means, both $u_{2\sigma}^2$ and $u_{2\sigma_{C_2}}^2$ becomes permanent cycles. By the decomposition $u_{2\rho_4} = u_{2\sigma}^2 N_2^4(u_{2\sigma_{C_2}})$, we see that $u_{2\rho_4}$ is not permanent cycle but $u_{2\rho_4}^2$ is. Thus the integral degree cycle is $(\bar{r}_1^{C_4})^4 u_{2\rho_4}^2$ in degree $16$. Therefore $D^{-1}MU^{((C_4))})^{hC_4}$ is also $8$-periodic.*
(2) *In the above example, if we instead invert $N(\bar{r}_3^{C_2})$ and $N(\bar{r}_1^{C_4})$, we then make $u_{2\sigma}^2$ and $u_{2\sigma_{C_2}}^4$ permanent cycles. That means $u_{2\rho_4}^4$ is the minimal permanent cycle after inverting. The integral graded permanent cycle is $(\bar{r}_1^{C_4})^8 u_{2\rho_4}^4$ in degree $32$.*

From these examples we see that by choosing different $\bar{r}_{2^i-1}^H$ to invert for different $H \subset G$, we can obtain various different periodic homotopy fixed point. But there is only one spectrum we want.

**Corollary 3.3.** *Let $G = C_8$ and $D = N(\bar{r}_{15}^{C_2}\bar{r}_3^{C_4}\bar{r}_1^{C_8})$. Then in $\tilde{\Omega} := D^{-1}MU^{((C_8))}$, $u_{2\rho_8}^{16}$ is a permanent cycle, and in integral degree $N(\bar{r}_1^{C_8})^{32}u_{2\rho_8}^{16}$ is a cycle in degree $256$. Therefore $\Omega := (\Omega)^{hC_8}$ is $256$-periodic.*

The spectrum $\Omega$ is the detecting spectrum used in proving the non-existence of Kervaire invariant one elements.

**Remark 3.4.** *As one can see, the choice of elements to invert affects the periodicity in an essential way, and we can produce lower periodicity by inverting smaller $\bar{r}$. The only reason we choose $G = C_8$ and $D = N(\bar{r}_{15}^{C_2}\bar{r}_3^{C_4}\bar{r}_1^{C_8})$ is that this is the smallest setting that the detection theorem works, which is an incredible combination of equivariant homotopy, chromatic homotopy theory and tools from local class field theory.*

Before we end this part, there is some foundation to mention. If we are careful enough, we see that we are using norm in an essential way in $D^{-1}MU^{((C_8))}$ to prove that its homotopy fixed point is $256$-periodic. But in general, it is NOT ture that localization preserves $G$-commutative ring structure, therefore those norms are not guaranteed. The following theorem of Hill and Hopkins makes sure that in this case everything works out.

**Theorem 3.5.** *Let $R$ be a $G$-commutative ring spectrum and $D \in \pi_\bigstar^G R$. If for each $H \subset D$, $N_H^G i_H^* D$ divides a power of $D$, then $D^{-1}R$ has a unique $G$-commutative ring structure such that $R \to D^{-1}R$ is a map of $G$-commutative rings.*

One can check that $N(\bar{r}_i)$ satisfies the condition: $N_H^G i_H^* N(\bar{r}_i) = N(\bar{r}_i)^{[G:H]}$, therefore we do have norms in $D^{-1}MU^{((C_8))}$.

## 4. THE HOMOTOPY FIXED POINT THEOREM

Besides of the detection theorem, there is still one more step to patch everything together: The gap theorem tells us that in the fixed point $\pi_{-2}(D^{-1}MU^{((C_8))})^{C_8} = \underline{0}$, but the periodicity theorem tells us that the homotopy fixed point $(D^{-1}MU^{((C_8))})^{hC_8}$ is $256$-periodic. One commonly used comparison machinery between fixed points and homotopy fixed points is the Tate diagram. Let $EG$ be a contractible free $G$-CW complex and $X$, then we can consider the following diagram of $G$-spectra, namely, the Tate diagram of $X$.

$$\begin{tikzcd}
EG_+ \wedge X \arrow[r] \arrow[d, "\simeq"] & X \arrow[r] \arrow[d] & \widetilde{EG} \wedge X \arrow[d] \\
EG_+ \wedge F(EG_+, X) \arrow[r] & F(EG_+, X) \arrow[r] & \widetilde{EG} \wedge F(EG_+, X)
\end{tikzcd}$$

The horizontal maps are induced by the cofibre sequence $EG_+ \to S^0 \to \widetilde{EG}$ smashing with either $X$ or $F(EG_+, X)$. The vertical map is induced by applying $F(-x)$ to $EG_+ \to S^0$. Notice first that the left vertical map is an equivalence, since both spectra are free. Therefore when $X = D^{-1}MU^{((C_8))}$, if we can show both $\widetilde{EG} \wedge X$ and $\widetilde{EG} \wedge F(EG_+, X)$ are contractible, then by five lemma on the long exact sequence of homotopy groups, $X^{C_8} \simeq X^{hC_8}$.

First we show that $\widetilde{EG} \wedge X$ is contractible. We can do it by proving all geometric fixed points $\Phi^H(\widetilde{EG} \wedge X$ is contractible. When $H = \{e\}$, since the underlying space of $\widetilde{EG}$ is $S^\infty$, it is contractible. For nontrivial $H \subset G$, since $\Phi^H$ is monoidal, we have

$$\Phi^H(\widetilde{EG} \wedge X \simeq \Phi^H(\widetilde{EG}) \wedge \Phi^H(X) \simeq S^0 \wedge \Phi^H(X) \simeq \Phi^H(X).$$

Now since some $\overline{r}^H_{2^i-1}$ is invertible for each nontrivial $H$, and $\Phi^H(N^H_{C_2}(\overline{r}^H_{2^i-1})) = 0$, $\Phi^H(X) = 0$. For $F(EG_+, X)$, notice that it is a module over $X$, therefore $\Phi^H(F(EG_+, X))$ is a retract of $\Phi^H(F(EG_+, X) \wedge X)$, which is contractible. This finishes the proof.

**Remark 4.1.** *The idea of the proof works for any ring spectrum in the weakest sense: Given a $G$-homotopy ring spectrum $R$, if you can show that for any nontrivial $H \subset G$, $\Phi^H(R)$ is contractible, then $R^G \simeq R^{hG}$.*

## References

[HHR] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) **184** (2016), no. 1, 1–262. MR3505179