the Detection Theorem Preliminaries

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August 17, 2019

The goal of these two talks is to prove the detection theorem:

Theorem 1. *[2, Theorem 11.1] If $\theta_j \in \pi_{2^{j+1}-2}S^0$ is an element of Kervaire invariant* 1*, and $j > 2$, then the image of θ_j in $\pi_{2^{j+1}-2}\Omega$ is non-zero.*

We will use Adams-Novikov spectral sequence (ANSS) to study $\theta_j \in \pi_{2^{j+1}-2}S^0$ and its image in $\pi_{2^{j+1}-2}\Omega$. This will reduce the problem to an algebraic one on the E_2 page. The algebraic problem can be reduced to an easier one via a construction using formal A-modules. The goal of this talk is to briefly introduce ANSS and formal A-modules.

1 Adams-Novikov Spectral Sequence

1.1 Construction

Set up: let E be an associative ring spectrum (homotopy commutative) and assume that E_*E is flat over E_*.

Theorem 2. *[1, Theorem 15.1] Given a spectrum X, we have the E-based Adams-Novikov spectral sequence*

$$E_2^{s,t} = Ext^s_{E_*E}(E_*, E_{*+t}X) \Rightarrow \pi_{t-s}X_E^{\wedge}.$$

The construction follows from the cosimplicial resolution:

$$X_\bullet \colon X \to E \wedge X \to E \wedge E \wedge X \to \cdots .$$

The total space $Tot(X_\bullet)$ is $X_E^{\wedge}$ and the ANSS is the Bousfield-Kan spectral sequence.

1.2 E_2 page

Definition 3. *[5, A 1.1.1] A Hopf algebroid over a commutative ring K is a cogroupoid in the category of (graded or bigraded) commutative K-algebras,i.e., a pair (A, Γ) of commutative K-algebra with structure maps such that for any commutative K-algebra B, the sets $Hom(A, B)$ and $Hom(\Gamma, B)$ are the objects and morphisms of a groupoid.*

The pair (E_*, E_*E) is a Hopf algebroid. Here is the structure.

$$E \to E \wedge E \to E \wedge E \wedge E$$

where the first part has three maps: left unit, right unit, multiplication; the second map is id$\wedge$unit$\wedge$id. We can identify $E \wedge E \wedge E$ with $(E \wedge E) \underset{E}{\wedge} (E \wedge E)$ (E is associative). Applying π_*, the second part gives the coproduct (E_*E is flat over E_*)

$$E_*E \to E_*E \underset{E_*}{\otimes} E_*E.$$

Similarly, E_*X is a E_*E comodule.

In general, if (A, Γ) is a Hopf algebroid and M is a Γ comodule, we have the cobar resolution

$$C^*_\Gamma \colon M \to \Gamma \underset{A}{\otimes} M \to \Gamma^{\otimes^2_A} \underset{A}{\otimes} M \to \cdots .$$

By definition, the cohomology $H^*(C^*_\Gamma)$ is $\mathrm{Ext}^*_{E_*E}(E_*, M)$. One can identify the E_2 page from this.

1.3 Examples

Example 4. *The classical Adams spectral sequence Let E be $H\mathbb{F}_2$, X be the sphere spectrum S. Then $S^\wedge_{H\mathbb{F}_2}$ is the 2-completed sphere $S^\wedge_2$ and*

$$E_*E = \mathcal{A} = \mathbb{F}_2[\xi_1, \xi_2, \cdots],\ |\xi_i| = 2^i - 1$$

is the dual Steenrod algebra.

Here are some interesting elements on the E_2 page $Ext_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. In this case, the cobar resolution to compute the E_2-page is

$$\mathbb{F}_2 \to \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \to \cdots .$$

In degree 1, we will write $\xi_1 \in \mathcal{A}$ as $[\xi_1]$. In degree 2, we will write $\xi_1 \otimes \xi_1 \in \mathcal{A} \otimes \mathcal{A}$ as $[\xi_1|\xi_1]$. You can see the pattern.

$$[\xi_1^{2^i}] = h_i \in Ext^1 \Rightarrow Hopf\ \ invariant\ 1\ elements.$$

. For example, h_0 converges to 2, h_1 converges to η, but from h_4, they do not survive in the homotopy.

$$d_2 h_i = h_0 h_{i-1}^2.$$

$$[\xi_1^{2^i}|\xi_1^{2^i}] = h_i^2 \in Ext^2 \Rightarrow Kervaire\ invariant\ 1\ elements.$$

Example 5. *The Adams-Novikov spectral sequence Let E be the p primary Brown-Peterson spectrum BP, X be the sphere spectrum S. Then $S^\wedge_{BP}$ is the p-local sphere $S_{(2)}$ and*

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots],\ |v_i| = 2(p^i - 1),$$

$$BP_*BP = BP_*[t_1, t_2, \cdots],\ |t_i| = 2(p^i - 1).$$

In this case, the cobar resolution to compute the E_2-page is

$$BP_* \to BP_*BP \otimes BP_* \to BP_*BP \otimes BP_*BP \otimes BP_* \to \cdots .$$

For elements in degree 1, for example $t_1 \otimes 1 \in BP_*BP \otimes BP_*$, we use the notation $[t_1]1$, sometimes we omit the 1 and write $[t_1]$ for this class. For elements in degree 1, for example $t_1^2 \otimes t_1 \otimes v_1 \in BP_*BP \otimes BP_*BP \otimes BP_*$, we use the notation $[t_1^2|t_1]v_1$ and you can see the pattern.

1.4 Thom Reduction

There is map from ANSS to ASS induced by the map of Hopf Algebroid

$$(BP_*, BP_*BP) \to (H\mathbb{F}_2, \mathcal{A})$$

where $BP_* \to H\mathbb{F}_2$ is quotient map by $(2, v_1, \cdots)$ and $BP_*BP \to \mathcal{A}$ sends t_i to $\bar{\xi}^2$.

Example 6. *Under Thom reduction, $[t_1]$ goes to $[\bar{\xi}_1^2] =[\xi_1^2]$.*

1.5 Greek letter elements

The structure map in (BP_*, BP_*BP) is much more complicated. One can refer to [5, Theorem A2.1.27] for the explicit formulas if one would like to try the computation by hands. To have better understanding of the E_2 page, we introduce the greek letter elements. We will write $\mathrm{Ext}_{BP_*BP}(BP_*, M)$ as $\mathrm{Ext}(M)$ for short.

Definition 7. *[5, Section 1.3] Denote the idea $(p, v_1, \cdots , v_{n-1}) \subset BP_*$ by I_n. The short exact sequence*

$$0 \to BP_*/I_n^\infty \to BP_*/I_{n-1}^\infty[v_{n-1}^{-1}] \to BP_*/I_{n-1}^\infty \to 0$$

induces a long exact sequence in Ext *groups. Denote the boundary map by*

$$\delta_n\colon\ \mathrm{Ext}^s(BP_*/I_n^\infty) \to \mathrm{Ext}^{s+1}(BP_*/I_{n-1}^\infty).$$

Suppose that $x = v_n^{i_n}/(p^{i_0}, v_1^{i_1}), \cdots , v_{n-1}^{i_{n-1}} + \cdots$ (we only write the leading term) is an element in $\mathrm{Ext}^0(BP_*/I_n^\infty)$, then

$$\alpha^{(n)}_{i_n/i_{n-1},\cdots,i_0} := \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n x$$

where $\alpha^{(n)}$ is the n^{th} Greek letter. When $i_0 = 1$, we omit it from the notation.

Example 8. *α family and β family.*

$$\beta_{i/j} \in \mathrm{Ext}^{2,6i-2j}_{BP_*BP}(BP_*, BP_*).$$

Fact 9. *In the bidegree of θ_j, there is only one class h_j^2 in the classical Adams spectral sequence. However, in the ANSS, there are more than one elements. For example, in the bidegree of θ_4, there are $\beta_{8/8}$, $\beta_{6/2}$ and $\alpha_1\alpha_{2^j-1}$ ([6, Section 7.1]).*

The first two lines of the ANSS have been studied in [4]. We follow notations in [6] to describe classes in ANSS in the bidegree of θ_j as follows:

$$\beta_{c(j,k)/2^{j-1-2k}};$$

$$\alpha_1\alpha_{2^j-1}$$

where $0 \leqslant k < j$ *and* $c(j,k) = 2^{j-1-2k}(1+2^{2k+1})/3$.

2 Formal A-modules

Follow the notations in [2, Section 11.2]. Let A and R be commutative rings, and $e\colon A \to R$ a ring homomorphism.

Definition 10. *[5, A2.1.1] A (commutative 1-dimensional) formal group law over R is a power series $F(x,y) \in R[[x,y]]$ satisfying*

1. $F(x,0) = F(0,x) = x$,
2. $F(x,y) = F(y,x)$,
3. $F(x,F(y,z)) = F(F(x,y),z)$.

Definition 11. *[5, A2.1.5] Let F and G be formal group laws. A homomorphism from F to G is a power series $f(x) \in R[[x]]$ with constant term 0 such that*

$$f(F(x,y)) = G(f(x),f(y)).$$

It is an isomorphism if it is invertible, i.e., if $f'(0)$ (the coefficient of x) is a unit in R, and a strict isomorphism if $f'(0) = 1$. A strict isomorphism from F to the addition formal group law $x+y$ is a logarithm for F, denoted by $log_F(x)$.

We have a ring homomorphism

$$\mathbb{Z} \to \mathrm{End}(F)$$

$$n \to [n](x)$$

where $[1](x) = x$ and $[n+1](x) = F(x,[n](x))$.

Definition 12. *A formal A-module over R is a formal group law F over R, equipped with a ring homomorphism*

$$A \to \mathrm{End}(F)$$

$$a \to [a](x)$$

with the property that $[a]'(0) = e(a)$.

We are interested in the case $A = \mathbb{Z}_2[\zeta]$ where ζ is a primitive 8^{th} root of unity and $R_* = A[u, u^{-1}]$ where $|u| = 2$. The maximal ideal of A is generated by $\pi = \zeta - 1$. A is a discrete valuation ring. The valuation can be given by the divisibility of π. For example, 2 is π^4·unit in A.

Given a power series $f(x) \in A[[x]]$ such that

$$f(x) = \pi x \bmod (x^2)$$

$$f(x) = x^2 \bmod (\pi),$$

Lubin and Tate's work [3] constructed a formal A-module F_f over A (unique up to isomorphism) such that

$$[\pi](x) = f(x).$$

For $a \in A$, write

$$[a](x) = a_d x^d + \cdots \bmod (\pi)$$

with $0 \neq a_d \in A/(\pi)$ One can check that the function $v(a) = \log_q(d)$ defines a valuation on A. For example, $v(\pi) = 1$, $v(2) = 4$. We can define a homogeneous formal group law over a graded ring by setting $|x| = |y| = -2$. From a formal group law F_f, we can define a homogeneous formal group law F over R_* by

$$uF(x, y) = F_f(ux, uy).$$

3 Group actions and the ANSS [2, 11.3.2, 11.3.3]

Let $\mathcal{M}_{FG}$ be the category of pairs (R, F), with F a formal group law over a commutative ring R, and in which a morphism

$$(f, \psi)\colon (R_1, F_1) \to (R_2, F_2)$$

consists of a ring homomorphism $f\colon R_1 \to R_2$, and an isomorphism of formal group laws $\psi\colon F_2 \xrightarrow{\cong} f_* F_1$. A (left) action of a group on (R, F) is a map of monoids

$$G \to \mathcal{M}_{FG}((R, F), (R, F)).$$

We define a trivial $C_8 = \langle \gamma \rangle$ action on A. Then C_8 acts on the pair (A, F_f) by

$$f_\gamma\colon A \xrightarrow{\text{id}} A,$$

$$\psi_\gamma = [\zeta](x)\colon F_f \to f_* F_f = F_f.$$

The $C_8 = \langle \gamma \rangle$ action can be extended to (R_*, F) by

$$\gamma u = \zeta u.$$

Example 13. *[2, Example 11.18] Here is an example of (R, F) with G action. Suppose that E is a complex oriented, homotopy commutative ring spectrum, and that a finite group G acts on E by homotopy multiplicative maps. Let F denote the corresponding (homogeneous) formal group law over $\pi_* E$. Then the action of G on $E^*(\mathbb{C}P^\infty)$ gives an action of G on $(\pi_* E, F)$.*

One can associate a Hopf algebroid to a pair (R, F) with G action. Let $C(G, R_*)$ be the ring of maps (as set) from G to R_*. (In our case, G is C_8 and $R_* = \mathbb{Z}_2[\zeta][u, u^{-1}]$.) The pair $(R_*, C(G; R_*))$ is a Hopf algebroid. The structure maps are

$$\eta_L \colon R_* \to C(G; R_*)$$

sending $r \in R_*$ to the constant function with value r;

$$\eta_R \colon R_* \to C(G; R_*)$$

sending $r \in R_*$ to the function $g \to g \cdot r$;

$$\Delta \colon C(G; R_*) \to C(G; R_*) \underset{R_*}{\otimes} C(G; R_*),$$

the composition of the map

$$C(G; R_*) \to C(G \times G; R_*)$$

dual to multiplication in G, and the isomorphism

$$C(G; R_*) \underset{R_*}{\otimes} C(G; R_*) \xrightarrow{\cong} C(G \times G; R_*)$$

given by setting

$$(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1) \cdot g_1 f_2(g_2).$$

Moreover, there is a map of Hopf algebroids

$$(MU_*, MU_*MU) \to (R_*, C(G; R_*)) \tag{1}$$

where the map $MU_* \to R_*$ classifies the formal group law F, and the map $MU_*MU \to C(G, R_*)$ is defined by declaring the composition

$$MU_*MU \to C(G, R_*) \xrightarrow{\mathrm{ev}_g} R_*$$

to be the map classifying the strict isomorphism

$$[g](x) \colon F \to g^*F.$$

(Here we use the fact that MU_*MU represents strict isomorphism between formal group laws. A map $MU_*MU \to R$ is equivalent to a strict isomorphism between F_1 and F_2.)

The map 1 induces a map

$$\mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*) \to H^s(G; R_t). \tag{2}$$

When the G-action on (R_*, F) arises, as in Example 13, from an action of G on a complex oriented homotopy commutative ring spectrum E, the map 2 is the E_2-term of a map of spectral sequences abutting to the homomorphism $\pi_*S^0 \to \pi_*E^{hG}$(see details in [2, 11.3.3]).

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