# Cohomology Theories and Naive Spectra

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# 1 Introduction

We will review Eilenberg-Steenrod axioms for cohomology. We construct Eilenberg-MacLanes spaces, which represent cohomology theories. This leads to naive  $\Omega$ -spectra and Brown representability.

# 2 Axiomatic cohomology

In this section we review Eilenberg-Steenrod axioms for cohomology theories.

A cohomology theory E consists of  $\mathbb{Z}$ -graded contravariant functors  $E^n$ , from the category of pairs of CW complexes to the category of abelian groups, and natural transformations  $\delta : E^n(A) := E^n(A, \emptyset) \Rightarrow E^{n+1}(X, A)$ , such that:

• Exactness. The following sequence is exact:

$$\dots \to E^n(X,A) \to E^n(X) \to E^n(A) \to E^{n+1}(X,A) \to \dots$$

• **Homotopy.** If  $f : (X, A) \to (Y, B)$  is a homotopy equivalence, then

$$f^*: E^n(Y,B) \xrightarrow{\cong} E^n(X,A).$$

• Excision. If X is the union of subcomplexes A and B, then the inclusion  $(A, A \cap B) \rightarrow (X, B)$  induces an isomorphism

$$E^n(X,B) \xrightarrow{\simeq} E^n(A,A\cap B).$$

• Additivity. If  $(X, A) = \coprod_i (X_i, A_i)$ , then

$$E^n(X,A) \xrightarrow{\cong} \prod_i E^n(X_i,A_i)$$

Conventionally, if  $E^n(*) = 0$  for  $n \neq 0$ , we call E an *ordinary* cohomology theory.

There is a based variant: a reduced cohomology theory consists of  $\mathbb{Z}$ -graded functors  $\tilde{E}^n$ , from the category of based CW complexes to the category of abelian groups, and natural isomorphisms

$$\delta: \tilde{E}^n(X) \xrightarrow{\cong} \tilde{E}^{n+1}(\Sigma X),$$

such that:

• **Exactness.** If A is a subcomplex of X then the following sequence is exact:

$$\tilde{E}^n(X/A) \to \tilde{E}^n(X) \to \tilde{E}^n(A).$$

- Homotopy. If  $f \simeq g : X \to Y$  are based homotopic, then  $f^* = g^*$ .
- Wedge. If  $X = \bigvee_i X_i$ , then

$$\tilde{E}^n(X) \xrightarrow{\cong} \prod_i \tilde{E}^n(X_i).$$

In this context the dimension axiom reads as  $\tilde{E}^n(S^0) = 0$  for  $n \neq 0$ .

The relation between reduced and unreduced cohomology is the following:

$$\tilde{E}^*(X) = E^*(X,*), \quad E^*(X) = E^*(X_+), \quad E^*(X,A) = \tilde{E}^*(X/A).$$

**Example.** Cellular/singular cohomology theory HG. **Example.** Ordinary cohomology of  $S^n$ .

Cup product and homology.

## 3 Eilenberg-MacLane spaces

Given n > 0 and a discrete group G, the *Eilenberg-MacLane space* K(G, n), is characterized by the following property:  $\pi_n K(G, n) = G$ , while  $\pi_k K(G, n) = 0$  for  $k \neq n$ . Of course if n > 1 we require that G is abelian.

One way to construct Eilenberg-MacLanes spaces is by attaching cells. Say  $n \ge 1$ . Present G with generators and relations:

$$G = \langle g_1, ..., g_{\alpha} / r_1, ..., r_{\beta} \rangle.$$

The homotopy group  $\pi_n(\vee_i S_i^n)$  is free abelian with  $\alpha$  generators. Each relation  $r_i$  is represented by a based map  $S^n \to \vee_i S_i^n$ . One could attach a (n+1)-cell via this attaching map to realize the relation  $r_i$ . The result is a space X with trivial homotopy groups  $\pi_i(X)$  for i < n and  $\pi_n(X) = G$ .

The same method could be used to kill all higher homotopy groups. Starting with  $\pi_{n+1}$ , we attach (n+2)-cells via attaching maps  $S^{n+1} \to X$  that generate  $\pi_{n+1}$ , and this won't affect lower homotopy groups. This finishes the construction.

Eilenberg-MacLane spaces are unique up to weak homotopy equivalence, some examples are

$$K(\mathbb{Z},1)\simeq S^1, \quad K(\mathbb{Z}/2,1)\simeq \mathbb{R}P^\infty, \quad K(\mathbb{Z},2)\simeq \mathbb{C}P^\infty.$$

Eilenberg-MacLane spaces represent cohomology theories. Recall that [X, Y] denotes the set of based homotopy classes of maps between X and Y, and  $\pi_0 F(X, Y) = [X, Y]$ . The construction above reveals that Eilenberg-MacLane spaces are naturally based.

**Theorem.** For CW complexes X, abelian groups G, and integers  $n \ge 0$ , there are natural isomorphisms

$$H^{n}(X;G) \cong [X, K(G, n)].$$

It is not hard to prove that for any based space Z, the functor [-, Z] from based CW complexes to pointed sets satisfies **Homotopy**, **Exactness** and **Wedge** conditions given in the Eilenberg-Steenrod

axioms for reduced cohomology theory. For the functor to take value in Abelian groups, we have to impose more structures on Z, for example if Z is a double loop space. Milnor proved that the loop space of a CW complex has the homotopy type of a CW complex. Hence we have a homotopy equivalence

$$\widetilde{\sigma}_n: K(G, n) \to \Omega K(G, n+1).$$

By iterating, Eilenberg-MacLane spaces are infinity loop spaces.

An  $\Omega$ -spectrum is a sequence of based spaces  $E_n$ ,  $n \ge 0$ , and based weak homotopy equivalences  $\tilde{\sigma}: E_n \to \Omega E_{n+1}$ . For an abelian group G, the Eilenberg-MacLane spectrum is  $\{K(G, n), \tilde{\sigma_n}\}$ .

**Proposition.** Let  $E = \{E_n\}$  be an  $\Omega$ -spectrum. Define

$$\widetilde{E}^{n}(X) = \begin{cases} [X, E_{n}] & \text{if } n \ge 0\\ [X, \Omega^{-n}E_{0}] & \text{if } n < 0. \end{cases}$$
(3.1)

Then the functors  $\tilde{E}^n$  define a reduced cohomology theory on based CW complexes. We only need to verify the suspension isomorphism, which is induced my  $\tilde{\sigma}$ :

$$\widetilde{E}^n(X) = [X, E_n] \to [X, \Omega E_{n+1}] \cong [\Sigma X, E_{n+1}] = \widetilde{E}^{n+1}(\Sigma X).$$

Now we have proved the theorem.

Cohomology could as well be generated to the  $\infty$ -categorical setting. The idea is that: given an  $\infty$ -category  $\mathbb{C}$ . For two objects X, A of  $\mathbb{C}$ , the *degree 0 cohomology of X with coefficients in A*, is the set of connected components of the hom space  $\mathbb{C}(X, A)$ .

We shall see Eilenberg-Maclane spaces also produce ordinary homology theories. By adjunction  $[\Sigma X, Y] \cong [X, \Omega Y]$ ,  $\tilde{\sigma}_n : K(G, n) \to \Omega K(G, n+1)$  corresponds to map

$$\sigma_n: \Sigma K(G, n) \to K(G, n+1)$$

We may smash with a based CW complex X to obtain

$$\pi_{n+k}(X \wedge K(G, n))) \xrightarrow{\Sigma} \pi_{n+k+1}(X \wedge \Sigma K(G, n)) \xrightarrow{(\mathsf{Id} \wedge \sigma_n)_*} \pi_{n+k+1}(X \wedge K(G, n+1)).$$

**Theorem.** For based CW complexex X, abelian groups G, and integers  $n \ge 0$ , there are natural isomorphisms

$$\tilde{H}_k(X,G) \cong \operatorname{colim}_n \pi_{n+k}(X \wedge K(G,n)).$$

A spectrum is a sequence of based spaces  $E_n$ ,  $n \ge 0$ , and based maps  $\sigma_n : \Sigma E_n \to E_{n+1}$ . Given nice conditions, one expect similar results. But we won't go into details here. You will see an example at the beginning of next talk.

Now we build the Eilenberg-MacLane spaces into the construction of Postnikov towers which can be expressed as tower of fibrations with Eilenberg-MacLane spaces as fibers. We say a topological space is *n*-truncated if the homotopy groups of X vanish in dimensions larger than n. Recall that the Postnikov tower of path-connected X, is a sequence of spaces

$$X \to ... \to X_n \xrightarrow{p_n} X_{n-1} ... \to X_1 \xrightarrow{p_1} X_0$$

such that

(1).  $\pi_i(X_n) \cong \pi_i(X)$  for  $i \leq n$ ,

(2).  $X_n$  is *n*-truncated, i.e.,  $\pi_i(X_n) = 0$  for i > n.

We could construct a Postnikov tower by attaching cells when X is a CW complex. The Postnikov tower, if it exists, is unique up to homotopy.

Furthermore, one could successively replace each map  $p_n$  by a fibration: given a map  $f: X \to Y$ , define the *path space*  $Nf = X \times_f Y^I$ . *Nf* consists of pairs  $(x, \gamma)$  such that  $f(x) = \gamma(0)$ . Now *f* could be decomposed as

$$X \xrightarrow{\nu} Nf \xrightarrow{p} Y$$
,

where  $\nu(x) = (x, \gamma_{f(x)})$  and  $\rho(x, \gamma) = \gamma(1)$ . It is not hard to check that *Nf* deformation retracts to *X* and *p* is a fibration.

By examining the homotopy long exact sequence, the new map  $p'_n$  is a fibration with fiber  $K(\pi_n(X), n)$ . One recovers the space X by taking the homotopy limit of the tower. This kind of tower resolution construction is both theoretically and computationally important.

### 4 Brown Representability

On the other hand, the representability of ordinary cohomology is a consequence of a general result called the Brown representability theorem.

Recall that if  $\mathcal{C}$  is a category and  $F : C^{\text{op}} \to \text{Set}$  is said to be *representable* if there exists  $X \in \mathcal{C}$  and an isomorphism  $F \to \text{Hom}_{\mathcal{C}}(-, X)$ .

There is a notion of *presentable* categories, as well as a notion of *presentable*  $\infty$ -categories.

**Proposition.** Let  $\mathcal{C}$  be a presentable category, and  $F : \mathcal{C}^{op} \to Set$  be a functor. F is representable if and only if F preserves limits.

**Proposition.** Let  $\mathcal{C}$  be a presentable  $\infty$ -category, and  $F : \mathcal{C}^{op} \to \mathcal{S}$  be a functor. F is representable if and only if F preserves small limits.

There are also nice criteria (Adjoint Functor Theorem) to determine whether a functor between presentable ( $\infty$ -)categories has left/right adjoints. As an example, the *n*-truncation functor is the left adjoint of the inclusion of  $\infty$ -category of *n*-trucated spaces into S.

A contravariant functor from the homotopy category of based connected CW complexes to the category of pointed sets is called a *Brown functor* if it satisfies the following conditions:

(1). it takes coproducts to products,

(2). it takes weak pushouts to weak pullbacks.

**Theorem.** (Brown representability) Brown functors are representable. Every reduced cohomology theory on the category of based CW complexes is represented by an  $\Omega$ -spectrum.

#### References

- J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659
- [2] J. Lurie, *Higher Algebra*, available at http://www.math.harvard.edu/~lurie/papers/HA.pdf.
- J. P. May, A concise course in algebraic topology, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999. MR1702278