the Detection Theorem

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August 17, 2019

The goal of this talk is to sketch the proof of the detection theorem:

Theorem 1. *[1, Theorem 11.1] If $\theta_j \in \pi_{2^{j+1}-2}S^0$ is an element of Kervaire invariant 1, and $j > 2$, then the image of θ_j in $\pi_{2^{j+1}-2}\Omega$ is non-zero.*

The spectral sequences reduce the problem to an algebraic one. We will prove the algebraic detection theorem.

Theorem 2. *([1, Theorem 11.2]) If*

$$x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*MU}(MU_*, MU_*)$$

is any element mapping to

$$h_j^2 \in \mathrm{Ext}^{2,2^{j+1}}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$$

in the E_2-term of the classical Adams spectral sequence, and $j > 2$, then the image of x in $H^2(C_8; \pi^u_{2^{j+1}}\Omega_{\mathbb{O}})$ is nonzero.

We follow the proof in [2], which is the proof in their preprint of August 26, 2009. The formal A-module constructions reduce the coefficients $\pi^u_{2^{j+1}}\Omega_{\mathbb{O}}$ to a much simpler ring $R_* = \mathbb{Z}_2[\zeta][u, u^{-1}]$. We show that the image of x as above is nonzero in the composition

$$\mathrm{Ext}^{2,2^{j+1}}_{MU_*MU}(MU_*, MU_*) \to H^2(C_8; \pi^u_{2^{j+1}}\Omega_{\mathbb{O}}) \to H^2(C_8; R_{2^{j+1}}), \tag{1}$$

which sends t_i to a R_*-valued function on C_8 determined by

$$[\xi](x) = \sum_{i \geqslant 0}^{F} t_i(\xi)(\xi x)^{2^i}. \tag{2}$$

1. Step 1: Construct maps in 1 and show that the composition is induced by the map of Hopf algebroids

$$(MU_*, MU_*MU) \to (R_*, C(C_8; R_*));$$

2. Step 2: Check the image of all such x in $H^2(C_8; R_{2^{j+1}})$.

We have seen maps

$$\mathrm{Ext}^{2,2^{j+1}}_{MU_*MU}(MU_*, MU_*) \to H^2(C_8; \pi^u_{2^{j+1}}\Omega_{\mathbb{O}}),$$

$$\mathrm{Ext}^{2,2^{j+1}}_{MU_*MU}(MU_*, MU_*) \to H^2(C_8; R_*).$$

For step 1, we need to proof the following lemma.

Lemma 3. *[2, 7.3 Lemma 2] The classifying homomorphism for F*

$$\lambda\colon \pi_*(MU) \to R_*$$

factors through $\pi_*(MU^{(4)}_{\mathbb{R}})$ in such a way that

1. *the homomorphism $\lambda^{(4)}\colon \pi_*(MU^{(4)}_{\mathbb{R}}) \to R_*$ is equivariant;*
2. *the element $D \in \pi_*(MU^{(4)}_{\mathbb{R}})$ that we invert to get $\Omega_{\mathbb{O}}$ goes to a unit in R_*.*

For the first statement, a homomorphism $\lambda^{(4)}\colon \pi_*(MU^{(4)}_{\mathbb{R}}) \to R_*$ is equivalent to the data

$$F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} F_3 \xrightarrow{f_3} F_4$$

where F_i are formal group laws over R_* and f_i are strict isomorphisms. Recall the C_8 action on $MU^{(4)}_{\mathbb{R}}$ is given by C_4 permutation and the C_2 action on $MU_{\mathbb{R}}$. An C_8 equivariant map is equivalent to a formal $\mathbb{Z}[\zeta]$-module structure on F_i. We start with a formal $\mathbb{Z}_2[\zeta]$ module, hence we have the equivariant map.

The second part needs explicit computation. We will come back to it later. A remark is that the choice of D is the smallest one such that its image is a unit in R_*.

Before going into the tricky computation in step 2, we start with a warm up in the baby case:

Example 4. *We will do a detection theorem: KO detects the first Hopf invariant one element η. Here is the baby algebraic detection theorem:*

Claim 5. *If*

$$x \in \mathrm{Ext}^{1,2}_{MU_*MU}(MU_*, MU_*)$$

is any element mapping to

$$h_1 \in \mathrm{Ext}^{1,2}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$$

in the E_2-term of the classical Adams spectral sequence, then the image of x in $H^2(C_2; \pi_*KU)$ is nonzero.

We will work at prime 2. One can replace MU by BP, KU by $KU^{\wedge}_2$. One should work with 2-typical formal group laws over $\mathbb{Z}_{(2)}$. Here we are not very careful to these details (one can show the coefficient of x^2 in $[-1]$ is the same for the multiplicative formal group law and its 2-typicalization). Recall that on the E_2 page *of ANSS, in the degree of η, there is only one nontrivial class α_1, and under the Thom reduction $\alpha_1 \sim t_1 \to \xi_1^2 \sim h_1$. The map*

$$\mathrm{Ext}_{\mathcal{A}}^{1,2}(\mathbb{Z}/2, \mathbb{Z}/2) \to H^1(C_2, \pi_2 KU)$$

is induced by the map of Hopf algebroid

$$(MU_*, MU_*MU) \to (KU_*, C(C_2, KU_*)),$$

where the map $MU_* \to KU_*$ classifies the multiplicative formal group law $F(x, y) = x + y + uxy$ over $KU_* = \mathbb{Z}[u, u^{-1}]$. Let γ be the generator of C_2. γ acts on F by $[-1]_F(x)$. Then the image of t_1 sends γ to the coefficients of x^{2^1} in $[-1]_F(x)$. From

$$F(x, [-1]_F(x)) = 0,$$

one can solve that $[-1]_F(x) = -x + ux^2 + \cdots$ and $t_1(\gamma)$ is a unit. So the image of t_1 is nontrivial in $C(C_2, KU_*)$. Recall that $H^*(C_2, KU_*)$ is the cohomology of the complex (cobar resolution)

$$KU_* \to C(C_2, KU_*) \to C(C_2 \times C_2, KU_*) \to \cdots .$$

In inner degree 2, C_2 acts by -1 on KU_2, so the first map for $a \in KU_2$

$$a \to (\gamma\colon a \to a - \gamma a)$$

is multiplication by 2 ($\gamma a = -a$). If the image of t_1 is a unit in $C(C_2, KU_*)$, not divided by 2, then it will not be in the image of d_1 and represent a nonzero cohomology class. On the other hand, $H^*(C_2, KU_*)$ is the cohomology of the complex (minimal resolution)

$$KU_* \xrightarrow{\gamma-1} KU_* \xrightarrow{\gamma+1} KU_* \xrightarrow{\gamma-1} \cdots .$$

We have $H^1(C_2, KU_2) = KU_2/2 = \mathbb{Z}/2$. This shows that α maps to the nonzero element in $H^1(C_2, KU_2) = \mathbb{Z}/2$.

We now sketch the proof of step 2. Recall that on the E_2 page of ANSS, in the bidegree of θ_j, there are

$$\beta_{c(j,k)/2^{j-1-2k}};$$

$$\alpha_1 \alpha_{2^j - 1}$$

where $0 \leqslant k < j$ and $c(j, k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. Under the Thom reduction, everything goes to 0, except $\beta_{2^{j-1}/2^{j-1}}$ goes to h_j^2. Hence, the candidates of x is $\beta_{2^{j-1}/2^{j-1}}$ plus a linear combination of the rest classes. We will show that in $H^2(C_8; R_{2^{j+1}})$

1. $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image;

2. the rest goes to zero.

We compute $H^2(C_8; R_{2^{j+1}})$ as the cohomology of the cochain complex

$$R_*[C_8] \xrightarrow{\gamma-1} R_*[C_8] \xrightarrow{1+\gamma+\cdots+\gamma^7} R_*[C_8] \xrightarrow{\gamma-1} \cdots .$$

Recall the $C_8 = \langle\gamma\rangle$ action on $R_* = A[\xi]$ is given by

$$\gamma a = a \text{ for } a \in A,$$

$$\gamma u = \xi u.$$

When $j \geqslant 3$, C_8 acts trivially on $R_{2^{j+1}}$ and $H^2(C_8; R_{2^{j+1}}) = u^{2^j}A/(8)$. We can define a valuation on A by the divisibility of π. Note that $\pi^4 = 2{\cdot}$unit. If we assign $v(\pi) = 1$, then $v(2) = 4$. This can be extended to a valuation on R_* by setting $v(u) = 0$. To check the image of the rest goes to 0, it is enough to show their images have valuation greater than $v(8) = 12$.

We will fix the formal A-module F over $R_* = A[u, u^{-1}]$ with logarithm

$$\log_F(x) = \sum_{n\geqslant 0} \frac{u^{2^n-1}x^{2^n}}{\pi^n}.$$

This will give concrete formulas for the map of Hopf algebroids (the formula for v_n is basic algebra but very complicated)

$$(BP_*, BP_*BP) \to (R_*, C(C_8, R_*)).$$

In particular, v_n goes to $\pi^{4-n}u^{2^n-1}{\cdot}$unit for $1 \leqslant n \leqslant 4$. When $n \geqslant 4$, v_n goes to a unit. Hence, we can assign valuation $v(v_n)$ =max$(0, 4-n)$. It is doable but painful to find the cocycle of $\beta_{i/j}$. However, the valuation will not decrease in the process. Hence, we have a lower bound of the valuation from the leading term v_2^i/v_1^j. When $j \geqslant 6$, all $\beta_{c(j,k)/2^{j-1-2k}} \neq \beta_{2^{j-1}/2^{j-1}}$ have valuation at least 20. Therefore, they are zero in $u^{2^j}A/(8)$. Now we check the image of $\beta_{2^{j-1}/2^{j-1}}$. We calculate with BP-theory. Recall that

$$BP_*BP = BP_*[t_1, t_2, \cdots] \text{ where } |t_n| = 2(2^n-1).$$

It is known that $\beta_{2^{j-1}/2^{j-1}}$ is cohomologous to

$$b_{j-1} = \frac{1}{2}\sum_{0<i<2^j}\binom{2^j}{i}[t_1^i|t_1^{2^j-i}] \in \mathrm{Ext}^{2,2^{j+1}}.$$

The image of t_n under $BP_*BP \to C(C_8, R_*)$ can be solved from

$$[\zeta](x) = \sum_{n\geqslant 0}^{F} t_n(\zeta)x^{2n}.$$

In particular, the function t_1 sends a primitive root in C_8 to a unit in R_*. Further computation shows that b_{j-1} goes to $4u^{2j}{\cdot}$unit.

Remark 6. *The image of b_{j-1} is divided by 4 and b_{j-1} is of order 2. This is a hint that one wants to work with the group C_8.*

In the end, we come back to the second part of Lemma 3. The choice of D is based on computation of $\lambda^{(4)}(r^H_{2^k-1})$ for various H and k and then choose the smallest one. Here we list some results of $r^{C_4}_{2^k-1}$.

$$\lambda^{(4)}(r_1^{C_4}) = (-\pi - 2)u = \pi \cdot \text{unit} \cdot u$$

$$\lambda^{(4)}(r_3^{C_4}) = (8\pi^3 + 26\pi^2 + 25\pi - 1)u^3 = \text{unit} \cdot u^3$$

We see that $r_3^{C_4}$ is the first one that maps to a unit. The same happens for $r_{15}^{C_2}$ and the formula is very complicated.

$$\lambda^{(4)}(r_{15}^{C_2}) = (306347134\pi^3 - 3700320563\pi^2 - 15158766469\pi - 16204677587)u^{15}.$$

References

[1] Hill, Michael A., Michael J. Hopkins, and Douglas C. Ravenel. "On the nonexistence of elements of Kervaire invariant one." Annals of Mathematics (2016): 1-262.

[2] Ravenel, Douglas C. Equivariant stable homotopy theory and the Kervaire invariant. Draft available online: `http://web.math.rochester.edu/people/faculty/doug/mybooks/esht.pdf`