Equivariant Spaces

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1 G-spaces and G-CW complexes

The main objects in equivariant homology theory and homotopy theory are G-spaces which are spaces equipped with an action by a topological group G. In details, here is the definition.

Definition 1.1. A (left) G-space is a topological space X with continuous actions $G \times X \to X$ such that $ex = x$ and $g(g'x) = (gg')x$.

Definition 1.2. A G-map $f : X \to Y$ is a continuous map f such that $f(gx) = gf(x)$ (We call maps satisfying this by equivariant maps).

They togeother form the category of G-spaces, $G\mathrm{Top}$.

The usual constructions on spaces apply equally well in this category. In particular, we have Cartesian product $X \times Y$ with G acting diagonally: which means $g(x, y) = (gx, gy)$. The space of all continous maps from X to Y, $\mathrm{Map}(X, Y) = Y^X$ is a G-space too. The action is $(g \cdot f)(x) = gf(g^{-1}x)$.

Remark: $G\mathrm{Top}(X, Y) = \mathrm{Map}(X, Y)^G$, G-fixed points of the mapping space. We will introduce fixed points next section.

As mentioned for nonequivarint spaces, we take all spaces to be conpactly generated and weak Hausdorff, we have a G-homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

Be careful here, Map means all continuous maps from X to Y instead of the morphism in the category $G\mathrm{Top}$.

All the terminologies above have based version:

Definition 1.3. A (left) based G-space is a based topological space X with continuous actions $G \times X \to X$ such that $ex = x$, $g(g'x) = (gg')x$ and the basepoint fixed by G.

Definition 1.4. A G-map $f : X \to Y$ is a based continuous map f such that $f(gx) = gf(x)$ (We call maps satisfying this by equivariant maps).

We use $G\mathrm{Top}_*$ to denote this cateogry. Recall that in based category, smash is product and wedge is coproduct,

$$\mathrm{Map}_*(X \wedge Y, Z) \cong \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$$

They together form the category of based G-spaces, $G\mathrm{Top}_*$.

Remark: There is a functor $G\mathrm{Top} \to G\mathrm{Top}_*$ sending X to X_+ which is disjoint union with addition point $*$ with trivial G-action.

Analogy to spaces, we now introduced G-CW complex to approximate G-spaces.

Definition 1.5. A G-CW complex X is the union of sub G-spaces X^n such that X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching G-cells $G/H \times D^{n+1}$ along the attaching G-maps $G/H \times S^n \to X^n$.

If we recall the definition of CW complex introduced yesterday, compare the pushout diagram.

$$\begin{tikzcd} \coprod_\alpha S^n \arrow[r] \arrow[d, hook] & X^n \arrow[d] \\ \coprod_\alpha D^{n+1} \arrow[r] & X^{n+1} \end{tikzcd}$$

Remark: Compare the pushout diagrams of non-equivariant CW complex and of equivariant CW complex. To give a sense that G/H plays the role of points in nonequivariant case.

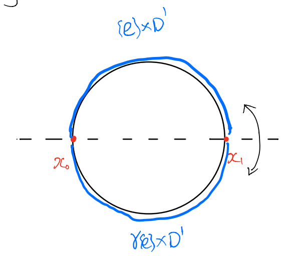
$$\begin{tikzcd} \coprod_\alpha G/H_\alpha \times S^n \arrow[r] \arrow[d, hook] & X^n \arrow[d] \\ \coprod_\alpha G/H_\alpha \times D^{n+1} \arrow[r] & X^{n+1} \end{tikzcd}$$

The attaching map $G/H \times S^n \to X^n$ is determined by its restriction $S^n \to (X^n)^H$.

In equivariant theory orbits, G/H play the role of points.

Example 1.6. Use the examples of S^1 with reflection action of $\mathbb{Z}/2$ and S^1 with antipodal action of $\mathbb{Z}/2$.

Eg 1.



S^1 with C_2-action by reflection with respect to the dotted line. Write $C_2 = \{e, \gamma\}$

note that the two red dots are fixed by C_2-action. We can take G-CW complex structure as:

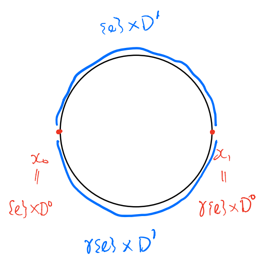
0-cells: $x_0 = C_2/C_2 \times D^0$, $x_1 = C_2/C_2 \times D^0$

1-cell: $C_2/\{e\} \times D^1$

The attaching map is by identifying $C_2/\{e\}\times\{0\}$ to x_0, $C_2/\{e\}\times\{1\}$ to x_1

$$\begin{tikzcd} C_2/\{e\}\times S^0 \arrow[r] \arrow[d] & X^0 = \{x_0, x_1\} \arrow[d] \\ C_2/\{e\}\times D^1 \arrow[r] & X^1 = S^1 \end{tikzcd}$$

Eg 2.



S^1 with C_2 acting as antipodal map.

Note: There is no fixed points.

We can take G-CW complex structure as:

0-cell: $C_2/\{e\} \times D^0$

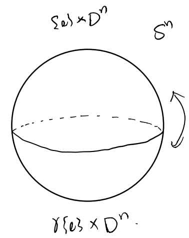
1-cell: $C_2/\{e\} \times D^1$

The attaching map is different from above

$$\begin{tikzcd} C_2/\{e\}\times S^0 \arrow[r] \arrow[d] & X^0 = \{x_0, x_1\} \arrow[d] \\ C_2/\{e\}\times D^1 \arrow[r] & X^1 = S^1 \end{tikzcd}$$

We identify. $\{e\}\times 0 \sim x_0$, $\{e\}\times 1 \sim x_1$
$\gamma\{e\}\times 0 \sim x_1$, $\gamma\{e\}\times 1 \sim x_0$.

Ex: S^n with C_2-action by reflection



Note the equator S^{n-1} is fixed

We can take $X^{n-1} = S^{n-1}$ with trivial action and cells of lower dimensions are

$C_2/C_2 \times$ cells of S^{n-1} in nonequivariant case.

n-cells: $C_2/\{e\} \times D^n$ w/ attaching map

$$\begin{tikzcd} C_2/\{e\} \times S^{n-1} \arrow[r] \arrow[d] & X^{n-1} = S^{n-1} \arrow[d] \\ C_2/\{e\} \times D^n \arrow[r] & X^n = S^n \end{tikzcd}$$

is by mapping $C_2/\{e\} \times S^{n-1} \longrightarrow S^{n-1}$

$(g\{e\}, x) \longmapsto x$.

2 Geometric fixed points and orbits

For $H \subset G$, we write the set of fixed points of H by $X^H = \{x | hx = x \text{ for } h \in H\} = \mathrm{Hom}_G(G/H, X)$ and Weyl group $W_G H = N_G H/H = \mathsf{Hom}_G(G/H, G/H)$. We denote the orbit space by $X_G = X/G$.

Consider the functor $F : \mathrm{Top} \to G\mathrm{Top}$, which sends a space Y to its underlying space with trivial G-action. F has both right and left adjoints, which are geometric fixed points and orbits. In particular,

$$G\mathsf{Top}(Y, X) \cong \mathsf{Top}(Y, X^G)$$

$$G\mathsf{Top}(X, Y) \cong \mathsf{Top}(X_G, Y)$$

In general, if we have a map of groups $G \to K$, it induces a functor $f^* : K\mathrm{Top} \to G\mathrm{Top}$. f^* has both left adjoint $f_!(X) = K \times_G X$ and right adjoint $f_*(X) = \mathrm{Map}(K, X)^G$. If we take $K = *$, it gives the above adjuntions.

Geometric fixed points and orbits are also limits and colimits. Let BG be the category with one object and G for morphisms. We can naively regard $G\mathrm{Top}$ as the (covariant) functor category $F(BG, \mathrm{Top})$. A map $f : G \to K$ induces a functor $F : BG \to BK$ and a functor $f^* : K\mathrm{Top} \to G\mathrm{Top}$, $f_!$ is the left Kan extension along F and f_* is the right Kan extension along F.

$$\begin{tikzcd} BG \arrow[r] \arrow[d, "F"'] & \mathrm{Top} \\ BK \arrow[ur, dotted, "f_!"'] \arrow[u, phantom, ""] & \end{tikzcd} \qquad \begin{tikzcd} BG \arrow[r] \arrow[d, "F"'] & \mathrm{Top} \\ BK \arrow[ur, dotted, "f_*"'] & \end{tikzcd}$$

The diagrams do not commute. There are natural transformation $\mathrm{id} \Rightarrow f^*f_!$ and $f^*f_* \Rightarrow \mathrm{id}$. In particular, if we take $K = *$, left and right Kan extensions of a functor X along F to the trivial category give the colimits and limits of X, we have

$$X^G = \lim_{BG} X.$$

$$X_G = \mathrm{colim}_{BG} X.$$

3 Homotopy fixed points and orbits

Now we define homotopy fixed points to be $G\mathrm{Top}(EG, X) = \mathsf{Map}(EG, X)^G := X^{hG}$ and homotopy orbits to be $EG \times_G X := X_{hG}$.

Remark 3.1. In the latter, we take X to be right G-space. (For a left G-space, we could take the action of g to be $g^{-1}\mathrm{x}$, it gives a right G-space structure.)

Recall from last section, a map $f : G \to K$ induces a map $f^* : K\mathrm{Top} \to G\mathrm{Top}$. We consider homootpy right Kan extension functor, the right derived functor Rf_* of the right Kan extension f_*. The idea of derived functor here is choosing a fibrant replacement functor $Q : \mathrm{Top}_G = \mathrm{Fun}(BG, \mathrm{Top}) \to \mathrm{Top}_G$ and setting $Rf_* = f_* \circ Q$. Taking $K = *$, it defines the homotopy limits:

$$\mathsf{holim}_{BG} X = X^{hG}.$$

By the properties of right derived functors, there is a natural map from a functor to its right derived functor. We get a natural map from limits to homotopy limits and in particular,

$$X^G \to X^{hG}.$$

Geometrically, $X = \mathrm{Map}(*, X) \to \mathrm{Map}(EG, X)$. Taking G-fixed points, it gives the map $X^G \to X^{hG}$, the same map as above. In fact, $EG \to *$ is nonequivariantly homotopy equivalence. The G-action of EG is free and the G-action of $*$ is trivial. Therefore, the map $X^G \to X^{hG}$ is nonequivariantly homotopy equivalence, but not equivariantly homotopy equivalence. (We will introduce equivariant homotopy next section)

If we do the entire process for left homotopy derived functors $Lf_!$, the left derived functor of left Kan extension $f_!$, we have

$$\mathsf{hocolim}_{BG} X = X_{hG}.$$

There is natural map from left derived functor to the functor and thus from homotopy colimts to colimts. We have a map

$$X_{hG} \to X/G.$$

Geometrically, it is the map $EG \times_G X \to * \times_G X = X/G$. It is nonequivariantly homotopy equivalence, but not equivariantly homotopy equivalence.

Remark 3.2. In $G\mathrm{Top}_*$, the homotopy fixed points and orbits are $\mathrm{Map}(EG_+, X)^G$ and $EG_+ \wedge_G X$.

4 Homotopies

Definition 4.1. A G-homotopy of f, g: $X \to Y$ is a G-map $H : I \times X \to Y$ with G acting trivially on I such that $H(0, x) = f(x)$ and $G(1, x) = g(x)$.

Definition 4.2. For a topological G-space X, $H \subset G$ a closed subgroup of G, its nth H-equivariant homotopy groups are

$$\pi_n^H(X) = \pi_0 \mathrm{Hom}_G(G/H_+ \wedge S^n, X) = \pi_n(X^H).$$

Definition 4.3. A G-map $f : X \to Y$ is weak homotopy equivalence if $f^H : X^h \to Y^H$ is a weak equivalence for all $H \subset G$.

Recall that in non-equivariant case, a map $f : Y \to Z$ is an n-equivalence if $\pi_p(f)$ is a bijection for $q < n$ and a surjection for $q = n$ (for any choice of basepoint). Now we give a analogous definition for equivariant case.

Definition 4.4. Let ν be a function from conjugacy classes of subgroups of G to the integers$\geqslant 1$. We say that a map $e : Y \to Z$ is a ν-equivalence if $e^H : Y^H \to Z^H$ is a $\nu(H)$-equivalence for all H.

Theorem 4.5 (Homotopy extension and lifting property)**.** *Let A be a subcomplex of a G-CW complex X of dimension ν and let $e; Y \to Z$ be a ν-equivalence. Suppose given maps $g : A \to Y$, $h : A \times I \to Z$, and $f : X \to Z$ such that $eg = hi_i$ and $fi = hi_0$ in the following diagram: then there exists maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commutes.*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll] & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dotted, "\tilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dotted, "\tilde{g}"]
\end{tikzcd}$$

Proof. We construct $\tilde{g}$ and $\tilde{h}$ on $A \cup X^n$ by induction on n. Passing from the n-skeleton to the $(n+1)$-skeleton, we may work one cell of X not in A at a time. By considering attaching mas, we quickly reduce the proof to the case when $(X, A) = (G/H \times D^{n+1}, G/H \times S^n)$ and this reduces to the nonequivariant case of (D^{n+1}, S^n) $\square$

Theorem 4.6 (Whitehead Theorem)**.** *Let $e : Y \to Z$ be a ν-equivalence and X be a G-CW complex. Then $e_* : \mathrm{Hom}_G(X, Y) \to \mathrm{Hom}_G(X, Z)$ is a bijection if X has dimension less than ν and a surjection if X has dimension ν.*

Proof. Apply Theorem 4.5 to $(X, \varnothing)$ for the surjectivity and to $(Z \times I, X \times \partial I)$ for the injectivity. $\square$

Corollary 4.7. *If $e : Y \to Z$ is a ν-equivalence between G-CW complexes of dimension less than ν, then e is a G-homotopy equivalence.*

Analogous to nonequivariant case, we have

Theorem 4.8. *For any G-space X, there is a G CW complex ΓX and a (G)-weak homotopy equivalence $\Gamma X \to X$.*

References

[EHCT] J. P. May. Equivariant Homotopy and Cohomology Theory.