Talk 2: Presheaves on Orbit Category

Jia (Hana) Kong

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We have discussed the category of G-space GTop in 2.1. In this note, we will look at another model of GTop, i.e. the category of presheaves on the orbit category that takes value in the category of spaces. Throughout this note, G is a finite group or a compact Lie group, and by saying the subgroups of G , we refer to only those closed subgroups.

For a group G, its orbit category Orb_G is the category whose objects are orbits G/H, and morphisms G-equivariant maps. A presheaf on orbit category is a contravariant functor from the orbit category Orb_G. Given a G-space X, take its H-fixed point for each subgroup H and the system of fixed point spaces $\{X^H, H \subset G\}$ forms a presheaf on orbit category. In the other direction, given a presheaf $\mathcal{X}$, evaluating at the orbit G/e with G action, we obtain a G-space $\mathcal{X}(G/e)$. This actually gives a pair of adjoint functors

$$\Theta\colon \mathcal{P}(\mathrm{Orb}_G) \rightleftarrows \mathrm{GTop} : \Phi,$$

where $\mathcal{P}(\mathrm{Orb}_G)$ denotes the category of presheaves on the orbit category that takes value in the category of spaces.

A. D. Elmendorf proved in [ELM] that this adjoint pair induces equivalence between the homotopy categories. In fact, with the correct model category structures on both hand sides, this equivalence of homotopy categories can be seen in the model categorical level:

Theorem 0.1. *(Elmendorf)*
There is a Quillen equivalence

$$\Theta\colon \mathcal{P}(Orb_G) \rightleftarrows GTop : \Phi.$$

1 Model category

The model category theory is for doing homotopy theory. Quillen developed the definition of a model category to formalize the similarities between homotopy theory and homological algebra.

1.1 Weak factorization system and model structure

Definition 1.1. A weak factorization system (WFS) on a category $\mathcal{C}$ is a pair $\mathcal{L}, \mathcal{R}$ of classes of morphisms of $\mathcal{C}$ such that

- Every morphism $f : X \to Y$ of $\mathcal{C}$ may be factored as the composition of a morphism in $\mathcal{L}$ followed by one in $\mathcal{R}$

$$f : X \xrightarrow{\in \mathcal{L}} Z \xrightarrow{\in \mathcal{R}} Y.$$

- The classes are closed under having the lifting property against each other:
  - $\mathcal{L}$ is precisely the class of morphisms having the left lifting property against every morphism in $\mathcal{R}$;
  - $\mathcal{R}$ is precisely the class of morphisms having the right lifting property against every morphism in $\mathcal{L}$.

Definition 1.2. A model structure on a category $\mathcal{C}$ is a choice of three distinguished classes of morphisms: cofibrations C, fibrations F and weak equivalences W, satisfying the following conditions:

- W contains all isomorphisms and is closed under two-out-of-three: given a composable pair of morphisms f, g, if two out of the three morphisms f, g, $g \circ f$ are in W, so is the third;
- $(C, F \cap W)$ and $(C \cap W, F)$ are two weak factorization systems on $\mathcal{C}$. $F \cap W$ is called acyclic fibrations and $C \cap W$ is called acyclic cofibrations.

When a category $\mathcal{C}$ is complete and cocomplete category with a model structure, we call it a model category.

A key example which motivated the definition is the category of topological spaces.

Example 1.3. The category of topological spaces, Top, admits a standard model category structure with fibrations as Serre fibrations, equivalences as weak homotopy equivalences and cofibrations as the retracts of relative cell complexes.

In fact the model structure is cofibrantly generated in the sense that there are small sets of morphisms I and J which permit the small object argument such that I generates C and J generates $C \cap W$ by taking transfinite composition of pushouts of coproducts and taking retracts. The generating sets are

$$I = \{S^{n-1} \to D^n, n \geqslant 0\}, \quad J = \{D^n \to D^n \times I, n \geqslant 0\}.$$

Example 1.4. Cofibrantly generated model structures transfer along adjunctions([HIR, 11.3.2]). Use the cofibrantly generated model structure on Top, we obtain a cofibrantly generated model structure on GTop with generating sets

$$I_{\mathrm{GTop}} = \{G/H \times i\}_{H \subset G, i \in I}, \quad J_{\mathrm{GTop}} = \{G/H \times j\}_{H \subset G, j \in J}.$$

Similarly, we can also put a cofibrantly generated model structure on $\mathcal{P}(\mathrm{Orb}_G)$ which has the generating sets:

$$I_{\mathcal{P}(\mathrm{Orb}_G)} = \{\mathrm{Map}_G(G/H, -) \times i\}_{H \subset G, i \in I},$$
$$J_{\mathcal{P}(\mathrm{Orb}_G)} = \{\mathrm{Map}_G(G/H, -) \times j\}_{H \subset G, j \in J}.$$

1.2 Homotopy category of model category

The homotopy category $Ho(\mathcal{C})$ of a model category $\mathcal{C}$ is the localization of $\mathcal{C}$ with respect to the class of weak equivalences

$$\mathcal{C} \to Ho(\mathcal{C}) = \mathcal{C}[W^{-1}],$$

so that the homotopy category has the universal property that the weak equivalences become actual isomorphisms.

Remark 1.5. This definition of homotopy category does not depend on the choice of fibrations and cofibrations. It only depends on the underlying category with weak equivalences $(\mathcal{C}, W)$. However, the model structure makes the homotopy category easier to handle. In fact, with a model structure, $Ho(\mathcal{C})$ is equivalent to the category whose objects are those which are both fibrant and cofibrant, and morphisms are the equivalence classes of morphism under left homotopy. This definition of homotopy category avoids the set theory technical issues one may meet with while doing localization.

1.3 Quillen equivalences

Quillen equivalences are one convenient notion of morphisms between model categories.

Definition 1.6. For $\mathcal{C}$ and $\mathcal{D}$ two model categories, an adjoint pair

$$F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is a Quillen adjunction if the following equivalent conditions are satisfied:

1. F preserves cofibrations and acyclic cofibrations;
2. G preserves fibrations and acyclic fibrations;
3. F preserves cofibrations and G preserves fibrations;
4. F preserves acyclic cofibrations and G preserves acyclic fibrations.

Definition 1.7. The Quillen adjoint pair

$$F \colon C \rightleftarrows D : G$$

is a Quillen equivalence, if for any cofibrant object $X \in C$ and fibrant object $Y \in D$, $FX \to Y$ is a weak equivalence iff the adjoint $X \to GY$ is a weak equivalence.

Proposition 1.8. *When (F, G) is an Quillen equivalence, they induces equivalence on homotopy categories, i.e. the derived functors (LF, RG) are equivalences of categories.*

2 Presheaves on orbit category

Definition 2.1. Orbit category associated to a group G, denoted by Orb_G, is the category whose

- objects are G-orbits G/H,
- morphisms are G-equivariant maps.

Example 2.2. Let GTop be the category of G-spaces and let $\mathcal{P}(\mathrm{Orb}_G)$ be the category of presheaves on the orbit category Orb_G. Let X be a G-space. Consider the fixed point functor $X^{(-)}$ for every subgroup $H \subset G$. This defines a functor $\Phi : \mathrm{GTop} \to \mathcal{P}(\mathrm{Orb}_G)$. Φ has a left adjoint $\Theta : \mathcal{P}(\mathrm{Orb}_G) \to \mathrm{GTop}$ which sends a presheaf χ to $\chi(G/e)$, on which the G action is induced by the group action on orbit G/e.

In the previous example, we get a adjoint pair

$$\Theta \colon \mathcal{P}(\mathrm{Orb}_G) \rightleftarrows \mathrm{GTop} : \Phi.$$

Consider the model structures on GTop and $\mathcal{P}(\mathrm{Orb}_G)$ which are inherited from the classical Quillen model structure on Top. It is easy to check that this is a Quillen adjunction, and the next question is that if it is a Quillen equivalence. And Elmendorf's theorem gives an affirmative answer to this question.

Theorem 2.3. *(Elmendorf)*
There is a Quillen equivalence

$$\Theta \colon \mathcal{P}(Orb_G) \rightleftarrows GTop : \Phi.$$

Elmendorf's original proof [ELM] only showed these two categories have the same homotopy theory by constructing explicitly a functor $\Psi : \mathcal{P}(\mathrm{Orb}_G) \to \mathrm{GTop}$ and a natural transformation $\epsilon : \Phi\Psi \to \mathrm{id}$ such that $\epsilon^H : (\Psi\chi)^H \to \chi(G/H)$ is a homotopy equivalence. We will also give another proof in the next section which proves the statement in the model category level.

3 Proof of Elmendorf's theorem

We provide two proofs here. One is by Piacenza and uses model category. The other is Elmendorf's original proof, by constructing explicitly the functor Ψ using bar construction.

3.1 Sketch proof using model category

[EHCT, VI.6]
To talk about Quillen equivalence, we need model category structures on both categories. The model structure we use on $\mathcal{P}(\mathrm{Orb}_G)$ and GTop are inherited by the classical model structure on the category of spaces, i.e. the weak equivalences and fibrations are defined level-wise, and the cofibrations are defined by lifting property. More precisely:

Definition 3.1. There is a model structure on $\mathcal{P}(\text{Orb}_G)$, where $f : \mathcal{X} \to \mathcal{Y}$ is a weak equivalence (resp. fibration) if $f(G/H) : \mathcal{X}(G/H) \to \mathcal{Y}(G/H)$ is a weak equivalence (resp. fibration) in Top. for every $G/H \in \text{Orb}_G$. $f : \mathcal{X} \to \mathcal{Y}$ is a cofibration if it has left lifting property with respect to all acyclic fibrations.

Definition 3.2. There is a model structure on GTop, where $f : X \to Y$ is a weak equivalence (resp. fibration) if $f^H : X^H \to Y^H$ is a weak equivalence (resp. fibration) in Top. for every subgroup $H \subset G$. $f : X \to Y$ is a cofibration if it has left lifting property with respect to all acyclic fibrations.

Under these model structures, we are able to identify the cofibrant objects in $\mathcal{P}(\text{Orb}_G)$: cofibrant objects are retracts of cellular objects, and cellular objects are generated under pushout along inclusions and direct colimits, by

$$\{\text{Map}_G(G/H, -) \times X| \;\; H \subset G, X \text{ is a cell in Top}\}$$

A key observation is that $(-)^H$ preserves retracts, relevant pushouts, and direct colimits. Therefore, by checking it on the generating objects, we obtain the following lemma:

Lemma 3.3. *If $\mathcal{X} \in \mathcal{P}(Orb_G)$ is cofibrant, then the unit of the adjunction*

$$\eta : \mathcal{X} \to \Phi\Theta(\mathcal{X})$$

is an isomorphism.

Proof. It's sufficient to check when $\mathcal{X} = \text{Map}_G(G/H, -) \times \mathcal{Y}$. And we have that

$$\mathcal{X}(G/K) = \text{Map}_G(G/H, G/K) \times \mathcal{Y} \cong (G/H)^K \times \mathcal{Y} \cong (\mathcal{X}(G/e))^K.$$

■

The lemma leads to the following proof of the main theorem.

Proof. (Elmendorf's theorem) Given $f : \Theta(\mathcal{X}) \to Y$, we have that

$$\mathcal{X}(G/H) \xrightarrow{\eta_H} (\mathcal{X}(G/e))^H \xrightarrow{f^H} Y^H.$$

By the previous lemma, η_H is a weak equivalence. The 2-out-of-3 axiom then shows that f^H is a weak equivalence iff the composition $\mathcal{X}(G/H) \to Y^H$ is. Since weak equivalences on $\mathcal{P}(\text{Orb}_G)$ and GTop are defined level-wise, $f : \Theta(\mathcal{X}) \to Y$ is a weak equivalence iff its adjoint $\mathcal{X} \to \Phi(Y)$ is. ■

3.2 Construction using bar construction

Since (Θ, Φ) is a Quillen equivalence, it induces a equivalence of homotopy categories, i.e. the derived functors $(L\Theta, R\Phi)$ are equivalences of homotopy categories. In the original proof in [ELM], the left derived functor $\Psi = L\Theta$ is constructed explicitly using bar construction.[EHCT, V.3]

Definition 3.4. Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a category equipped with a Cartesian product $\times$. Let F be a contravariant functor $\mathcal{C} \to \mathcal{D}$ and let G be a covariant functor $\mathcal{C} \to \mathcal{D}$. The construction $B_*(F, \mathcal{C}, G)$ defines a simplicial $\mathcal{D}$ object whose n-th level is

$$\amalg_{c_n \to \ldots \to c_0} F(c_0) \times G(c_n).$$

The face maps are defined by composition and the degeneracies by inserting the identity map.

When $\mathcal{D}$ is tensored over Top, we can take the geometric realization of this simplicial object, and we obtain the bar construction $|B_*(F, \mathcal{C}, G)|$.

By some general theory of bar construction, when G is a functor corepresented by $c \in \mathcal{C}$, i.e. $G = G_c = Map_{\mathcal{C}}(c, -)$, there is a natural homotopy equivalence obtained by composing and apply F:

$$|B_*(F, \mathcal{C}, G)| \to F(c).$$

Now we apply the above facts in our case. Let $\mathcal{C}$ be Orb_G, F be X and let G be the functor $J : \mathrm{Orb}_G \to \mathrm{GTop}$ which realizes each orbit G/H to the corresponding G-space G/H. Since G acts on J in a way compatible with the face and degeneracy maps, the bar construction gives a G-space.

Define $\Psi : \mathcal{P}(\mathrm{Orb}_G) \to \mathrm{GTop}$ to be

$$\Psi(X) := |B_*(X, \mathrm{Orb}_G, J)|.$$

Bar construction commutes with fixed point functors. Therefore we have

$$(\Psi(X))^H \cong |B_*(X, \mathrm{Orb}_G, J^H)| \xrightarrow{\simeq} X(G/H).$$

In other words, Ψ gives a homotopy inverse of Φ.

4 Applications

4.1 Coefficient system and Bredon (co)homology

An application of Elmendorf's theorem is the construction of Eilenberg-MacLane G-spaces.

Let $h\mathrm{Orb}_G$ denote the homotopy category of orbit category.

Definition 4.1. A coefficient system M is a contravariant functor $M : h\mathrm{Orb}_G \to \mathrm{Ab}$.

M can be regarded as a continuous contravariant functor from Orb_G to Ab.

Let B be the classifying space functor. Given any coefficient system M, we can regard the composition $B \circ M$ as a presheaf on the orbit category. Therefor we can construct Elenberg-MacLane space $K(M, 1) := \Psi(B \circ M)$. Similarly, we can construct $K(M, n) := \Psi(B^n \circ M)$.

With these Elenberg Maclane spaces, we can define a equivariant represented cohomology theory, i.e. the Bredon cohomology[EHCT, I.4], by taking

$$H^*(X; M) := [X, K(M, n)]_G.$$

It satisfies the Eilenberg-Steenrod axioms, with the classicale dimension axiom replaced by the following:

$$H^(G/H; M) \cong M(G/H).$$

In other words, in the equivariant world, orbits are thought of as points.

4.2 Universal space for a family of subgroups

Definition 4.2. $\mathcal{F}$ is a family of subgroups of G if it is a set of subgroups of G and it is closed under conjugation and taking subgroups.

We can change the morphism of weak equivalences on the category of GTop by taking

$$W = \{X \to Y | X^H \to Y^H \in W_{\text{Top}}, H \in \mathcal{F}\}.$$

We call it $\mathcal{F}$-equivalence.

Remark 4.3. When $\mathcal{F} = \{e\}$, $\mathcal{F}$ weak equivalence is just underlying equivalence. This corresponds to the category with only one object G/e and morphism indexed over G acting on it. It is the category denoted by BG. $\text{Fun}(BG, \text{Top})$ is category of G-spaces with the naive G-structure.

When $\mathcal{F}$ contains every subgroup so that the weak equivalences are as defined before, it corresponds to Orb_G. $\text{Fun}(\text{Orb}_G, \text{Top})$ is category of G-spaces with the genuine G-structure. It contains not only the data on G/e level, but also all the data on the subgroups, the transfers, restrictions and so on.

Definition 4.4. Let $\mathcal{F}$ be a family of subgroups of G. Define $E\mathcal{F}$ to be the presheaf where

$$E\mathcal{F}(G/H) := \begin{cases} *, & H \in \mathcal{F}, \\ \varnothing, & H \notin \mathcal{F}. \end{cases}$$

The universal space of $\mathcal{F}$ is defined to be the G-space $\Psi(\tilde{E}\mathcal{F})$. We abuse the notation and call the G-space $E\mathcal{F}$.

Remark 4.5. The universal spaces $E\mathcal{F}$ is $\mathcal{F}$-weakly contractible and $\mathcal{F}$-cofibrant. It has the property that if X has all its isotropy in $\mathcal{F}$, then $[X, E\mathcal{F}]_G \cong *$. When $\mathcal{F} = \{e\}$, this is the usual universal space EG, whose underlying space is weakly contractible.

Example 4.6. Consider the cofiber sequence

$$E\mathcal{F}_+ \to S^0 \to \tilde{E}\mathcal{F}.$$

This is a powerful tool called isotropy separation. For a G -space X. We smash it with the cofiber sequence, we will get

$$E\mathcal{F}_+ \wedge X \to X \to \tilde{E}\mathcal{F} \wedge X.$$

And it separate X into the $\mathcal{F}$-weak contractible space $\tilde{E}\mathcal{F} \wedge X$ and a $\mathcal{F}$-space $E\mathcal{F}_+$.

5 Universal property of presheaves and infinity categorical point of view

Let j denote the Yoneda embedding $j : D \to \mathcal{P}(D)$. Presheaves have the following universal properties[HTT, 5.1.5.6]:

Theorem 5.1. *There is an equivalence of infinity categories induced by Yoneda embedding j between*

$$\mathrm{Fun}^L(\mathcal{P}(D), \mathcal{C}) \to \mathrm{Fun}(D, \mathcal{C}),$$

where D is a simplicial set, $\mathcal{C}$ is an ∞-category which admits small colimits, and Fun^L denotes the full subcategory of functor category spanned by colimit preserving functors.

In other words, any functor $f : D \to \mathcal{C}$ equivalently factorizes as the composition $F \circ j$ where F is colimit-preserving.

We have the following infinity categorical version of Elmendorf's theorem.

Theorem 5.2. *(Elmendorf) The functor $\Phi : GTop \to \mathcal{P}(Orb_G)$ induces an equivalence of ∞-categories.*

Combine it with the following proposition:

Proposition 5.3. *The Yoneda embedding $j : D \to \mathcal{P}(D)$ generates $\mathcal{P}(D)$ under small colimits.*

We have that the analogous statement in the equivariant case:

GTop is generated by Orb_G under small colimits.

References

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