Application

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Let G be a compact Lie group. In this talk we are going to introduce the ordinary equivariant cohomology and the Smith theory as an application.

1 Ordinary Equivariant Cohomology

Definition 1.1. A coefficient system $\mathcal{A}$ is a contravariant functor $\mathcal{A} : h\mathcal{O}_G \to \mathbf{Ab}$.

Example 1.2. $\underline{\pi}_n(X)$, homotopy groups of a based G-space X: $\underline{\pi}_n(X)(G/H) = \pi_n(X^H)$.

This is an abelian category where kernels, cokernels, and biproducts are defined orbit wise. We shall define Bredon cohomology. First we give the axioms for the reduced theory, which determines the unreduced theory in the usual way.

Theorem 1.3. *Let G be a topological group, and $\mathcal{A}$ be a coefficient system. There exist (unique) functors*

$$\tilde{H}^n_G(-;\mathcal{A}) : hG\mathbf{Top}_*^{op} \to \mathbf{Ab}$$

together with isomorphisms $\sigma_n : \tilde{H}^n_G(X;\mathcal{A}) \cong \tilde{H}^{n+1}_G(\Sigma X;\mathcal{A})$. (where $n \in \mathbb{Z}$) satisfying the following axioms:

- *(Additivity) The inclusion $X_i \hookrightarrow \bigvee_i X_i$ induce an isomorphism:*

$$\tilde{H}^n_G(\bigvee_i X_i;\mathcal{A}) \cong \prod_i \tilde{H}^n_G(X_i;\mathcal{A}).$$

- *(Exactness) If $X \xrightarrow{f} Y \hookrightarrow Cf$ is a cofiber sequence, then the sequence*

$$\tilde{H}^n_G(Cf;\mathcal{A}) \to \tilde{H}^n_G(Y;\mathcal{A}) \xrightarrow{f^*} \tilde{H}^n_G(X;\mathcal{A})$$

  is exact.

- *(Weak equivalences) $\tilde{H}^n_G$ sends weak equivalences to isomorphisms.*
- *(Dimension) If G/H is an orbit, then*

$$\tilde{H}^n_G(G/H_+;\mathcal{A}) = \begin{cases} 0 & n \neq 0, \\ \mathcal{A}(G/H) & n = 0. \end{cases}$$

We shall not prove uniqueness here, but we will construct the Bredon cohomology for G-CW complexes.

Definition 1.4. Let X be a G-CW complex, $\mathcal{A}$ be a coefficient system. Define chain complex in the category of coefficient systems as follows:

$$\underline{C}_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H; \mathbb{Z})$$

The connecting homomorphism associated to the triple $((X^n)^H, (X^{n-1})^H, (X^{n-2})^H$ provides a map $d : \underline{C}_n(X) \to \underline{C}_{n-1}(X)$. We define a cochain complex of abelian groups as

$$C_G^n = \mathrm{Hom}_{\mathsf{Coeff}}(\underline{C}_n(X), \mathcal{A})$$

The homology of this complex is the Bredon cohomology, i.e. $H_G^*(X; \mathcal{A}) = HC_G^*$.

Example 1.5. If $\underline{A}$ is a constant coefficient system, then $H_G^*(X; \underline{A}) = H^*(X/G; A)$.

2 Smith Theory

Theorem 2.1 (P.A. Smith, 1939)**.** *Let G be a p-group, and X be a finite G-CW complex that is a mod p cohomology n-sphere. Then, X^G is either empty or a cohomology m-sphere, for some $m \leqslant n$. If p odd, then we have that $n - m$ is even, and if further n is even, X^G is non-empty.*

Remark 2.2. If $H \lhd G$, X^H is a natural G/H-space. Further, $X^G = (X^H)^{G/H}$.

Since G is a p-group, it is solvable. By the remark we need only consider the case $G = C_p$.

Proposition 2.3. *Let $G = C_p$. There are coefficient systems $\mathcal{A}, \mathcal{B}, \mathcal{C}$ so that:*

$$\begin{aligned} H_G^*(X; \mathcal{A}) &\cong H^*(X; \mathbb{F}_p) \\ H_G^*(X; \mathcal{B}) &\cong H^*(X^G; \mathbb{F}_p) \\ H_G^*(X; \mathcal{C}) &\cong \tilde{H}^*((X_+/X^G)/G; \mathbb{F}_p) \end{aligned}$$

Proof. We define:

$$\begin{array}{ll} \mathcal{A}(G) = \mathbb{F}_p[G] & \mathcal{A}(*) = \mathbb{F}_p \\ \mathcal{B}(G) = 0 & \mathcal{B}(*) = \mathbb{F}_p \\ \mathcal{C}(G) = \mathbb{F}_p & \mathcal{C}(*) = 0 \end{array}$$

We only need to verify the first four axioms of Bredon cohomology for the functors on the right, and then simply restrict these functors to the orbit category to compute $\mathcal{A}, \mathcal{B}, \mathcal{C}$. □

For the remainder of this section, all nonequivariant cohomology is taken in $\mathbb{F}_p$ coefficients.

Lemma 2.4. *Let $\mathcal{I}$ be the functor sending G to the augmentation ideal of $\mathbb{F}_p[G]$ and $*$ to 0, Then $\mathcal{I}^{p-1} = \mathcal{C}$, and we have exact sequences:*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \oplus \mathcal{C} \to 0$$

$$0 \to \mathcal{C} \to \mathcal{A} \to \mathcal{B} \oplus \mathcal{I} \to 0$$

and, for each $1 \leqslant n < p-1$,

$$0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to \mathcal{C} \to 0$$

We omit the verification of the lemma. The following theorem is the core of the proof.

Theorem 2.5. *We have the following equations:*

- $\sum_q \dim H^q(X^G) \leqslant \sum_q \dim H^q(X)$
- $\chi(X) = \chi(X^G) + p\tilde{\chi}((X_+/X^G)/G)$

Proof. From the first two sequences of Lemma 2.4, we have the following long exact sequences:

$$\cdots \to H^q_G(X;\mathcal{I}) \xrightarrow{\alpha} H^q(X) \xrightarrow{\beta} H^q(X^G) \oplus \tilde{H}^q((X_+/X^G)/G) \xrightarrow{\gamma} H^{q+1}_G(X;\mathcal{I}) \to \cdots$$

$$\tilde{H}^q((X_+/X^G)/G) \xrightarrow{\alpha'} H^q(X) \xrightarrow{\beta'} H^G(X^G) \oplus H^q_G(X;\mathcal{I}) \xrightarrow{\gamma'} \tilde{H}^{q+1}((X_+/X^G)/G)$$

We write $a_q = \dim H^q(X)$, $b_q = \dim H^q(X^G)$, $c_q = \dim \tilde{H}^q((X_+/X^G)/G)$, $i_q = \dim H^q_G(X;\mathcal{I})$. From the first sequence we have

$$i_{q+1} \geqslant \dim(\operatorname{Im}\gamma) = b_q + c_q - \dim(\operatorname{Im}\beta) = b_q + c_q - a_q + \dim(\operatorname{Im}\alpha) \geqslant b_q + c_q - a_q$$

$$\implies b_q + c_q \leqslant a_q + i_{q+1}$$

From the second, similarly we have

$$b_q + i_q \leqslant a_q + c_{q+1}$$

Adding these together, we have

$$2b_q + c_q + i_q \leqslant 2a_q + c_{q+1} + i_{q+1}$$

Summing over $0 \leqslant q \leqslant r_1$, and choosing r larger than the dimension of X (so that all cohomology at degree r or higher vanishes), we obtain:

$$2\sum b_q \leqslant c_0 + i_0 + 2\sum_q b_q \leqslant 2\sum_q a_q$$

as required.

To obtain the second equality, we quickly prove the following algebraic fact:

Proposition 2.6. *Consider the long exact sequence of finite dimensional vector spaces:*

$$\cdots \to C^{k-1} \xrightarrow{\gamma_{k-1}} A^k \xrightarrow{\alpha_k} B^k \xrightarrow{\beta_k} C^k \xrightarrow{\gamma_k} A^{k+1} \to \cdots$$

we have $\chi(B) = \chi(A) + \chi(C)$.

Proof. We have $\dim A^k = \dim(\operatorname{Im}\alpha_k) + \dim(\ker\alpha_k)$, and similarly for B, C. Using exactness, we write:

$$\chi(B) = \sum_k (-1)^k \dim(\ker\beta_k) + \sum_k (-1)^k \dim(\ker\gamma_k)$$
$$\chi(C) = \sum_k (-1)^k \dim(\ker\gamma_k) + \sum_k (-1)^k \dim(\ker\alpha_{k+1})$$
$$\chi(A) = \sum_k (-1)^k \dim(\ker\alpha_k) + \sum_k (-1)^k \dim(\ker\beta_k)$$

and therefore

$$\begin{aligned}\chi(B) - \chi(C) &= \sum_k (-1)^k \dim(\ker\beta_k) - \sum_k (-1)^k \dim(\ker\alpha_{k+1}) \\ &= \sum_k (-1)^k \dim(\ker\beta_k) + \sum_k (-1)6k \dim(\ker\alpha_k) = \chi(A)\end{aligned}$$

□

Returning to the proof of Theorem 2.5, we use either the first or second long exact sequence to write:

$$\chi(X) = \chi_{\mathcal{I}}(X) + \chi(X^G) + \tilde{\chi}((X_+/X^G)/G)$$

Using the third family of sequence in Lemma 2.4, we write:

$$\chi_{\mathcal{I}^n}(X) = \chi_{\mathcal{I}^{n+1}}(X) + \tilde{\chi}((X_+/X^G)/G)$$

Summing over $1 \leqslant n < p-1$, we have

$$\chi_{\mathcal{I}}(X) = \chi_{\mathcal{I}^{p-1}}(X) + (p-2)\tilde{\chi}((X_+/X^G)/G) = (p-1)\tilde{\chi}((X_+/X^G)/G)$$

Combining these expressions, we obtain:

$$\chi(X) = \chi(X^G) + p\tilde{\chi}((X_+/X^G)/G)$$

□

Proof of Theorem 2.1. If X is a cohomology sphere, then $\sum_q \dim H^q(X) = 2$. Then, by the first part of Theorem 2.5, $\sum_q \dim H^q(X^G)$ can be $0, 1$ or 2; by the second part, we can rule out 1, since the Euler characteristic of a cohomology sphere is either 0 or 2, and $\chi(X) \equiv \chi(X^G) \mod p$. If $\sum_q \dim H^q(X^G) = 0$, the X^G is empty. If $\sum_q \dim H^q(X^G) = 2$, then X^G is another cohomology sphere. If $p > 2$, then $\chi(S^n) \equiv \chi(S^m) \mod p$ iff $\chi(S^n) = \chi(S^m)$, so $n - m$ is even. If further n is even, then $\chi(X^G) \equiv 2 \mod p$, and so $X^G \neq 0$. □

References

[1] J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996, With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. MR1413302