Introduction to Equivariant Stable Homotopy Theory

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August 16, 2019

1 Motivation

Fix a compact Lie group G and a complete G-universe $\mathcal{U}$. The main goal of this talk is to give an introduction to equivariant stable homotopy theory and its point-set model $\mathbf{Sp}_{\mathbf{G}}^{\mathbf{O}}$, the category of orthogonal G-spectra. After the point-set model is built, we will introduce some basic constructions of orthogonal G-spectra while restricting to the case when G is a finite group.

In order to motivate the definition, let's first consider the equivariant Spanier-Whitehead category. The equivariant Spanier-Whitehead category $\mathcal{SW}^G$ has objects finite pointed G-CW complexes, and morphisms between objects X and Y are defined as the stablization in the universe $\mathcal{U}$:

$$\{X, Y\}_G := \mathrm{colim}_{V \subset \mathcal{U}}[\Sigma^V X, \Sigma^V Y]_G$$

as we have seen in the last talk, if the group G is finite then this is an finitely generated abelian group.

There is also an equivariant Spanier-Whitehead duality analogous to the non-equivariant one we have already seen on the first day. If we want formal dual in the sense of duality in closed symmetric monoidal category, then we need to add virtual representation spheres S^{-V} for all finite dimensional representation V to the Spanier-Whitehead category $\mathcal{SW}^G$. Now if we embed a G-CW complex X to a representation sphere S^V, then there is a G-CW complex Y and $S^{-V} \wedge Y$ is the dual of X in the enlarged Spanier-Whitehead category.

However, the category $\mathcal{SW}^G$ has some deficiencies to do stable homotopy theory, for example, it is neither complete or cocomplete.

In the following, we are going to construct a category $\mathbf{Sp}_{\mathbf{G}}^{\mathbf{O}}$ so that it (at least) satisfies all of the following properties:

- $\mathbf{Sp}_{\mathbf{G}}^{\mathbf{O}}$ is a bicomplete closed symmetric monoidal category
- The equivariant Spanier-Whitehead category $\mathcal{SW}^G$ embedds fully-faithfully into the homotopy category of $\mathbf{Sp}_{\mathbf{G}}^{\mathbf{O}}$.
- After specifying the weak equivalences in $\mathbf{Sp}_{\mathbf{G}}^{\mathbf{O}}$, the homotopy category of $\mathbf{Sp}_{\mathbf{G}}^{\mathbf{O}}$ is a triangulated category analogous to the stable homotopy category.

2 The category $\mathbf{Sp}_{\mathbf{G}}^{\mathbf{O}}$ of orthogonal G-spectra

The main references of this section are [GHT] and [Sch]. Let G be a compact Lie group. We will use $\mathbf{Sp}_{\mathbf{G}}^{\mathbf{O}}$ to denote the category of orhogonal G-spectra, as a point-set model for equivariant stable homotopy theory. Since we have already seen the definition of orthogonal spectra on the first day, we will also follow [GHT] for the definition of orthogonal G-spectra.

Let's first recall the definition from the first day. Denote $\mathbf{L}(V, W)$ the set of all linear embeddings for a pair of finite dimensional real inner product spaces V and W. Define $\xi(V, W) := \{(w, \phi) \in W \times \mathbf{L}(V, W) \mid w \in \phi(V)^{\perp}\}$ the "complementary" vector bundle over $\mathbf{L}(V, W)$. We denote $\mathbf{O}(V, W)$ as the Thom space of this bundle, which is the morhpism class of an indexing category $\mathbf{O}$. The objects of $\mathbf{O}$ are finite dimensional real inner product spaces. Note that if $\dim(W) < \dim(V)$, then $\mathbf{O}(V, W)$ is just a point. If $\dim(W) > \dim(V)$, then $\mathbf{O}(V, W)$ is a wedge of spheres $S^{W-\phi(V)}$ one for each linear embedding $\phi : V \to W$. If $\dim(W) = \dim(V)$, then $\mathbf{O}(V, W) = \mathbf{L}(V, W)_+$. An orthogonal spectrum is a based continuous functor from $\mathbf{O}$ and $\mathbf{Top}_*$.

Definition 2.1. [GHT][Definition 3.1.7] Let $\mathbf{Top}_*^G$ denote the category of based G-spaces with equivariant morphisms. An orthogonal G-spectrum is a based continuous functor from $\mathbf{O}$ to $\mathbf{Top}_*^G$. We denote the category of orthogonal G-spectra as $\mathbf{Sp}_{\mathbf{G}}^{\mathbf{O}}$.

Let's unpack this definition. An orthogonal G-spectrum assigns every real inner product space V a G-space $X(V)$. For each pair of inner product spaces V and W, we have an equivariant structure map $\mathbf{O}(V, W) \wedge X(V) \to X(W)$, which combines the $O(V)$-action on $X(V)$ and the structure maps $\sigma_{V,W} : S^{W-V} \wedge X(V) \to X(W)$ for every embedding $V \hookrightarrow W$. Moreover, all the compatibility conditions are packed in the functoriality.

Remark 2.2. As every n-dimensional real inner product space V is isomorhpic to a Euclidean space $\mathbb{R}^n$. The indexing category $\mathbf{O}$ has a small skeleton. We can recover an isomorphic but more explicit definition of orthogonal G-specta as in [Sch][Definition 2.1]. An orthogonal G-spectrum X is determined by the following data:

- a based space X_n with a based $O(n) \times G$ action for each $n \geqslant 0$,
- a based G-equivariant structure map $\sigma_n : X_n \wedge S^1 \to X_{n+1}$ for each $n \geqslant 0$, where G acts trivially on S^1,
- For all $m, n \geqslant 0$, the iterated structure map:

$$S^m \wedge X_n \xrightarrow{S^{m-1} \wedge \sigma_n} S^{m-1} \wedge X_{1+n} \xrightarrow{S^{m-2} \wedge \sigma_{n-1}} \cdots \xrightarrow{\sigma_{m-1+n}} X_{m+n}$$

  is $O(m) \times O(n)$-equivariant.

Remark 2.3. We should also remark that the above definition is not the same as those given in [HHR]. In [HHR], they define the indexing category $\mathbf{O}_G$ with objects all finite dimensional G-representations and with morphism $\mathbf{O}_G(V, W) = \mathbf{O}(V, W)$ as a based space equipped with a G-action by conjugation. Denote $\mathbf{Top}_G$ the topological G-category of G-spaces with non-equivariant maps. An orthogonal G-spectrum [HHR][Definition A.13] is a based continuous, enriched functor from $\mathbf{O}_G$ to $\mathbf{Top}_G$.

However, these two definitions are equivalent (as explained in [Sch][Remark 2.7]). Given an n-dimensional G-representation V and an orthogonal G-spectrum X, we can use our definition to recover its value at V as:

$$X(V) := \mathbf{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$$

where $O(n)$ acts on the right of $\mathbf{L}(\mathbb{R}^n, V)$ by precomposition and $X(V)$ is a G-space via diagonal action, i.e.

$$g \cdot [\phi, x] := [g\phi, gx]$$

for $\phi \in \mathbf{L}(\mathbb{R}^n, V)$ and $x \in X_n$. We leave it as an exercise to write down the general structure maps.

We mention again that we are working over a complete G-universe $\mathcal{U}$. This is important for the homotopy theory of genuine G-spectra.

The category $\mathbf{Sp}^{\mathbf{O}}_{\mathbf{G}}$ is tensored and cotensored over $\mathbf{Top}^{\mathbf{G}}$, which means that we can smash and take function object of a G-space and a G-spectrum levelwise. That is, if A is a G-space and X is a G-spectrum, then we can define G-spectra $A \wedge X$ and $\mathrm{Map}(A, X)$ levelwise, i.e:

$$(A \wedge X)(V) := A \wedge X(V)$$

where G acts diagonally, and

$$\mathrm{Map}(A, X)(V) := \mathrm{Map}(A, X(V))$$

where G acts by conjugation.

Notation 2.4. Denote $\Sigma^V X := S^V \wedge X$ and $\Omega^V X := \mathrm{Map}(S^V, X)$.

Definition 2.5. A homotopy between two maps $f, g : E \to F$ of orthogonal G-spectra is a map of orthogonal G-spectra $H : E \wedge I_+ \to F$ such that for each $V \in \mathcal{U}$, $H_{V,0} = f_V$ and $H_{V,1} = g_V$.

Given a based G-space A, we can define its suspension spectrum $\Sigma^\infty A$ with value at a G-representation V as $\Sigma^\infty A(V) := S^V \wedge A$.

Now let's denote $[E, F]_G$ the set of homotopy classes of maps from E to F. Then it's not hard to check that one has a fully-faithful embedding of $\mathcal{SW}^G$ into $\mathbf{Sp}^{\mathbf{O}}_{\mathbf{G}}$, i.e.:

$$[\Sigma^\infty X, \Sigma^\infty Y]_G \cong \mathrm{colim}_{V \subset \mathcal{U}} [\Sigma^V X, \Sigma^V Y]_G = \{X, Y\}_G$$

for each pair of finite G-CW complexes X and Y.

Next we will define the equivariant stable homotopy groups of orthogonal G-spectra associated to the complete G-universe $\mathcal{U}$.

Definition 2.6. Let X be an orthogonal G-spectrum. For each closed subgroup H of G, the H-equivariant 0-th stable homotopy group of X is defined as:

$$\pi_0^H(X) := \mathrm{colim}_{V\subset\mathcal{U}}[S^V, X(V)]_H$$

where $[-,-]_H$ denotes the homotopy classes of H-equivariant maps.

For k a positive number, we define the H-equivariant k-th stable homotopy group of X

$$\pi_k^H(X) := \mathrm{colim}_{V\subset\mathcal{U}}[S^{V\oplus\mathbb{R}^k}, X(V)]_H$$

and

$$\pi_{-k}^H(X) := \mathrm{colim}_{V\subset\mathcal{U}}[S^V, X(V\oplus\mathbb{R}^k)]_H$$

Definition 2.7. A map $f : X \to Y$ of orthogonal G-spectra is a π_*-isomorphism if it induces isomorphisms on homotopy groups:

$$\pi_n^H(f) : \pi_n^H(X) \to \pi_n^H(Y)$$

for all closed subgroups H and all integer n.

We will now define the smash product on the category $\mathbf{Sp_G^O}$ of orthogonal G-spectra. We first recall how the smash product is defined for orthogonal spectra. The smash product of orthogonal G-spectra in our setup is simply the smash product of the underlying non-equivariant orthogonoal spectra with diagonal G-action.

Let $\mathcal{V}$ be a closed symmetric monoidal category and $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ be a $\mathcal{V}$-enriched symmetric monoidal category, then there is a closed symmetric monoidal structure on the functor category $\mathcal{V}^{\mathcal{C}}$. The tensor product is given by Day convolution and the unit object is given by the corepresented functor $\underline{V}(1_{\mathcal{C}}, -)$. Equivalently, the Day convolution is equivalent to the left Kan extension of the external tensor product, defined by:

$$\begin{aligned} \bar{\otimes} : \mathcal{V}^{\mathcal{C}} \times \mathcal{V}^{\mathcal{C}} &\to \mathcal{V}^{\mathcal{C}\times\mathcal{C}} \\ (X, Y) &\mapsto \otimes_{\mathcal{V}} \circ (X, Y) \end{aligned}$$

along the tensor product $\otimes_{\mathcal{C}} : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$. In our case, we apply the Day convolution by letting $\mathcal{C} = \mathbf{O}$ and $V = \mathbf{Top}_*$, we then obtain the smash product in the category of orthogonal spectra $\mathbf{Sp}^O$.

Since our definition of orthogonal G-spectra are really orthogonal spectra with G-action, the smash product in $\mathbf{Sp_G^O}$ is the smash product in the underlying non-equivariant orthogonal spectra with diagonal action. Hence the universal property also follows:

$$\mathbf{Sp_G^O}(X \wedge Y, Z) \cong \mathrm{Bimor}((X, Y), Z)$$

where $\mathrm{Bimor}((X, Y), Z)$ denotes the class of bimorphisms from (X, Y) to Z.

As a special case of the Day's theorem [GHT][Theorem C.10], we obtain the closed symmetric monoidal structure on $\mathbf{Sp_G^O}$.

Theorem 2.8. *The category $\mathbf{Sp_G^O}$ of orthogonal G-spectra is a closed symmetric monoidal category with monoidal product the smash product $\wedge$, and unit object the equivariant sphere spectrum $\mathbb{S}$.*

3 Wirthmüller Isomorphism and Transfers

In this section, we assume the group G is a finite group. We will give the construction of the transfer map, induced and coinduced G-spectra and state the Wirthmüller isomorphism theorem. The main reference is [Sch][Section 4]. For those readers who are interested in the case with G being a compact Lie group, please refer to [GHT][Theorem 3.2.15].

3.1 Constructions of the transfer map

Given subgroups $K \subseteq H \subseteq G$, we have a natural map of left cosets $G/K \to G/H$ given by projection. This map lives in the category $\mathbf{Top}^G$ of G-spaces. The transfer map is a "wrong way" map $G/H \to G/K$ that lives in $\mathbf{Sp^O_G}$, that is, a map in $\mathrm{colim}_{V\subset\mathcal{U}}[\Sigma^V_+ G/H, \Sigma^V_+ G/K]_G$.

Let's choose an H-representation W and an H-equivariant embedding

$$j : H/K \hookrightarrow W$$

such an embedding is completely determined by the image $w := j(H)$. Without loss of generality, we can assume the open unit balls around the image points $g \cdot w$ are pairwise disjoint. Therefore, we get an embedding:

$$H/K_+ \wedge D(W) \to W$$

The Pontryagin-Thom collaspse map (i.e. sending the complement of the open balls $j(H/K_+ \wedge D(W))$ to the point at infinity) gives us a map:

$$S^W \to H/K_+ \wedge S^W$$

then compose with $G_+ \wedge_H (-)$, we get the desired transfer map:

$$tr^H_K : G/H_+ \wedge S^W \to G/K_+ \wedge S^W$$

which represents a map in the equivariant Spanier-Whitehead category $\mathcal{SW}^G$. Note that in the construction of the transfer tr^H_K, we made certain choice of the embedding $H/K \hookrightarrow W$. We should remark that different choices would lead to the same class in the Spanier-Whitehead category $\mathcal{SW}^G$.

Example 3.1. Let $G = H = C_3$ and $K = \{e\}$ be the trivial subgroup. We let W be a C_3-representation of $\mathbb{C}$ by a rotation of $\frac{2}{3}\pi$ around 0. We embed $C_3/\{e\}$ into W as 2, $2e^{\frac{2}{3}\pi i}$ and $2e^{\frac{4}{3}\pi i}$. Then the transfer map in this example is a map:

$$tr^{C_3}_{\{e\}} : S^W \to (C_3)_+ \wedge S^W$$

3.2 Induced and coinduced spectrum

Let H be a subgroup of G. The restriction functor $\mathrm{res}^G_H : \mathbf{Sp^O_G} \to \mathbf{Sp}^O_H$ has a left adjoint called induction and a right adjoint called coinduction, and both are given by space level construction $G_+ \wedge_H (-)$ and $\mathrm{Map}_H(G_+, -)$, where $\mathrm{Map}_H(-,-)$ denotes the H-equivariant maps.

Definition 3.2. Let X be a H-spectrum. The induced G-spectrum $G/H_+ \wedge X$ of X is defined for any G-representation V:

$$(G_+ \wedge_H X)(V) := G_+ \wedge_H X(i^*V)$$

where i^*V denotes the restriction of V to a H-representation.

The coinduced G-spectrum $F_H(G_+, X)$ of X is defined for any G-representation V:

$$F_H(G_+, X)(V) := \mathrm{Map}_H(G_+, X(i^*(V))$$

Exercise 3.3. Write down the structure maps of the induced and coinduced G-spectra of X.

3.3 Wirthmüller Isomorphism

Finally we are ready to state the Wirthmüller isomorphism.

Theorem 3.4 (Wirthmüller Isomorphism)**.** *[Sch][Theorem 4.9] Let H be a subgroup of G and X an H-spectrum. Then there is a π_*-isomorphism:*

$$G_+ \wedge_H X \simeq F_H(G_+, X)$$

Now let X in the above theorem be the sphere spectrum $\mathbb{S}$, we conclude that the orbits are self-dual.

Corollary 3.5 (Orbits are self-dual)**.** *The Spanier-Whitehead dual of G/H is itself.*

4 Derived Fixed points and Adams Isomorphism

4.1 Derived fixed points

Definition 4.1. Let X be an orthogonal G-spectrum. We define the naive fixed points spectrum X^G to be the level-wise G-fixed points of X with restricted $O(n)$-action. The structure map

$$\sigma_n^G : S^1 \wedge X_n^G \to X_{n+1}^G$$

is given by the restriction of the equivariant structure map $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ which makes sense because G acts on S^1 trivially.

The naive fixed points is not "homotopically correct", i.e. it doesn't send π_*-isomorphisms of orthogonal G-spectra to π_*-isomorphisms of orthogonal spectra. However, the naive fixed points functor can be right derived via replacing a spectrum X by a π_*-isomorphic G-Ω-spectrum, and then take naive fixed points.

For the purpose of this introduction, let's omit the technical model structure issues on $\mathbf{Sp}^{\mathbf{O}}_{\mathbf{G}}$ and assume we have the right derived functor of the naive fixed points functor denoted by $F^G : \mathbf{Sp}^{\mathbf{O}}_{\mathbf{G}} \to \mathbf{Sp}^{O}$.

Definition 4.2. Let X be an orthogonal G-spectrum. We call $F^G X$ the derived or genuine fixed point spectrum of X.

One shall see from the following proposition that the derived fixed points spectrum $F^G X$ capture the G-homotopy type of X.

Proposition 4.3. *For every orthogonal G-spectrum X and every integer k, we have isomorphisms:*

$$\pi_k^G(X) \cong \pi_k(F^G X)$$

4.2 Adams Isomorphism

Let X be a finite based G-CW-complex on which G acts cellularly and freely away from the based point. In the unstable world, the fixed point X^G is just a point. Adams made a surprising result in [Ad], stating that the G-fixed points of X is isomorphic to the G-orbits in the Spanier-Whitehead category $\mathcal{SW}^G$.

This result is a special case for a more general theorem. Let G be a finite group and N a normal subgroup. Denote $j : G \to G/N$ the projection and J the quotient group G/N. Let Y be a J-CW-complex which we consider as an object in the J-Spanier-Whitehead category $\mathcal{SW}^J$. Denote j^*Y the G-CW-complex with the induced G-action via j. Let X/N be the classical N-orbits of X, then X/N carries a natural J-action. The following is the original version of Adams isomorphism.

Theorem 4.4. *[Ad][Theorem 5.4] There is an isomorhpism:*

$$\{j^*Y, X\}_G \cong \{Y, X/N\}_J$$

Reich and Varisco [RV] lift this result into the category of orthogonal G-spectra before passing to stable homotopy category. Let $\mathcal{P}$ be a family of subgroups of G, we can construct a G-CW complex $E\mathcal{P}$ which is universal in the sense that $E\mathcal{P}^H$ is contractible whenever H is in $\mathcal{P}$ and $E\mathcal{P}^H$ is empty if H is not in $\mathcal{P}$.

Let $\mathcal{F}(N)$ be the family of subgroups H of G such that $H \cap N = \{1\}$.

Definition 4.5. An orthogonal G-spectrum X is good, if all the structure maps are closed embedding. An orthogonal G-spectrum X is N-free if the projection

$$E\mathcal{F}(N)_+ \wedge X \to X$$

is a π_*-isomorphism.

For any orthogonal G-spectrum X, they construct a natural map of orthogonal J-spectra, called the Adams map:

$$A : E\mathcal{F}(N)_+ \wedge_N X \to F^N X$$

Theorem 4.6. *[RV][Theorem 1.7] For any good, N-free, orthogonal G-spectrum. The Adams map A is a π_*-isomorphsim.*

5 Various Fixed Points Construction

In this section, we continue introducing fixed points constructions of orthogonal G-spectra. These fixed points constructions can be viewed as functors from the category $\mathbf{Sp^O_G}$ of orthogonal G-spectra to the category $\mathbf{Sp}^O$ of (non-equivariant) orthogonal spectra.

5.1 Geometric Fixed points and Isotropy Separation Sequence

Let's denote $\rho_G = \mathbb{R}[G]$ the regular representation of the finite group G. The geometric fixed points $\Phi^G X \in \mathbf{Sp}$ of a G-spectrum X is defined by:

$$\Phi^G X(V) := X(\rho_G \otimes V)^G$$

for a G-representation V, and with structure maps:

$$S^{W-V} \wedge \Phi^G X(V) \cong (S^{(W-V)\otimes\rho_G} \wedge X(\rho_G \otimes V))^G \xrightarrow{\sigma^G_{V,W}} X(\rho_G \otimes W)^G = \Phi^G X(W)$$

for every G-embedding $V \hookrightarrow W$.

Example 5.1. The geometric fixed points functor commutes with the suspension functor in the sense that the geometric fixed points $\Phi^G(\Sigma^\infty Y)$ is isomorphic to the suspension spectrum $\Sigma^\infty Y^G$ for any based G-space Y. This can be seen using the G-fixed points of the regular representation ρ_G is $\mathbb{R}$

$$\Phi^G(\Sigma^\infty Y)(V) = (Y \wedge S^{V\otimes\rho_G})^G \cong Y^G \wedge S^V = \Sigma^\infty Y^G(V)$$

for any G-representation V.

There is another way of constructing the geometric fixed point functor, for example as in [HHR]. Denote $\mathcal{P}$ the family of all proper subgroups of G. We can construct a G-CW complex $E\mathcal{P}$ which is universal in the sense that $E\mathcal{P}^H$ is contractible whenever H is a proper subgroup and $E\mathcal{P}^G$ is empty. For example, this can be done by taking $\operatorname{colim}_n S(n\bar{\rho_G})$, where $S(n\bar{\rho_G})$ is the unit sphere in $n\bar{\rho_G}$.

Definition 5.2. The isotropy separation sequence is the cofiber sequence:

$$E\mathcal{P}_+ \to S^0 \to \widetilde{E\mathcal{P}}$$

where the first map is sending $E\mathcal{P}$ to the non-based point. By the above definition of $\widetilde{E\mathcal{P}}$, we see that it can be characterized by the universal property that $\widetilde{E\mathcal{P}}^H \simeq S^0$ for any proper subgroup H and $\widetilde{E\mathcal{P}}^G \simeq *$.

In [HHR], the geometric fixed point of a G-spectrum X is defined to be the derived fixed point spectrum of $\widetilde{E\mathcal{P}} \wedge X$. These two definitions coincide for orthogonal G-spectra.

Proposition 5.3. *[Sch][Proposition 7.6] For any orthogonal G-spectrum X, we have a map of spectra called evaluation map*

$$ev : F^G(\widetilde{E\mathcal{P}} \wedge X) \to \Phi^G X$$

such that for any G-representation $W \in \mathcal{U}$, we have a weak equivalence $ev(W)$: $F^G(\tilde{E}\mathcal{P} \wedge X(W)) \to \Phi^G X(W)$.

So in particular, the above evaluation map is a π_*-isomorphism. Furthermore by Proposition 4.3, we see that the geometric fixed points functor is also homotopy invariant.

Let's wrap up the discussion of geometric fixed points by summarizing its properties, despite lacking the time to prove all of them:

Remark 5.4. The geometric fixed points functor $\Phi^G : \mathbf{Sp}^G_{\mathcal{U}} \to \mathbf{Sp}$ has the following properties

1. Φ^G is homotopy invariant which means it preserves π_*-isomorphism
2. Φ^G commutes with suspension, i.e. $\Phi^G \Sigma^\infty A^G \cong \Sigma^\infty A^G$ for any based G-CW complex A
3. Φ^G is symmetric monoidal
4. Φ^G commutes with filtered homotopy colimits

5.2 Homotopy Fixed points and Tate Construction

In this section, we'll see the homotopy fixed points, homotopy orbits and Tate constructions following [GM].

Recall that the fixed point functor $(-)^G$ in the category of G-spaces doesn't preserve weak equivalences. And we have seen that homotopy fixed points can actually detect (non-equivariant) weak equivalences of G-spaces. The homotopy fixed points of a G-space A is given by the space of G-equivariant maps $\mathrm{Map}_G(EG_+, A)$ and the homotopy orbits is given by its Borel construction $EG_+ \wedge_G A$.

Now we define the stable analogue of homotopy fixed points and homotopy orbits.

Definition 5.5. The homotopy fixed points X^{hG} of a G-spectrum X is defined as the (derived) fixed point spectrum $F^G(F(EG_+, X))$ and the homotopy orbits X_{hG} is defined as $EG_+ \wedge_G X$.

Denote $\widetilde{EG} := \mathrm{cofib}(EG_+ \to S^0)$. Since smashing with X preserves cofiber sequence, we have another cofiber sequence:

$$EG_+ \wedge X \to X \to \widetilde{EG} \wedge X$$

And by the isomorphism $X \cong F(S^0, X)$, we get a map $X \cong F(S^0, X) \to F(EG_+, X)$. This induces a diagram:

$$\begin{tikzcd}
EG_+ \wedge X \arrow[r] \arrow[d, "\simeq"] & X \arrow[r] \arrow[d] & \widetilde{EG} \wedge X \arrow[d] \\
EG_+ \wedge F(EG_+, X) \arrow[r] & F(EG_+, X) \arrow[r] & \widetilde{EG} \wedge F(EG_+, X)
\end{tikzcd}$$

It turns out that the left vertical map is always a π_*-isomorphism [GM][Proposition 1.2] and $F^G(EG_+ \wedge X) \simeq X_{hG}$ as a special case of the Adams isomorphism, hence after taking the (derived) fixed points F^G one has the following diagram

$$\begin{tikzcd}
X_{hG} \arrow[r] \arrow[d, "\simeq"] & F^G X \arrow[r] \arrow[d] & F^G(\widetilde{EG} \wedge X) \arrow[d] \\
X_{hG} \arrow[r, "\text{norm}"] & X^{hG} \arrow[r] & X^{tG}
\end{tikzcd}$$

where the right square is a pullback diagram and the lower right corner X^{tG} is called the **Tate spectrum** of X.

Moreover, if we take G to be a prime order cyclic group C_p, then $EG \simeq E\mathcal{P}$ and the top right corner becomes the geometric fixed point $\Phi^G X$ of X.

5.3 Tate Construction via ∞-category

We give a short explanation of the Tate construction in the modern ∞-categorical language following [NS]. The name comes from the following known fact: one should recover the Tate cohomology $\hat{H}^*(G; M)$ for a G-module M, when we compute the homotopy groups $\pi_*(HM^{tG})$ of the Tate spectrum of the Eilenberg-Maclane spectrum of M.

Let's fix a finite group G. Let $\mathcal{C}$ be an ∞-category and BG be the classifying space of G.

Definition 5.6. A G-equivariant object in $\mathcal{C}$ is a functor (i.e. a map of simplicial sets) from BG to $\mathcal{C}$. We denote $\mathcal{C}^{BG}$ the ∞-category of G-equivariant objects in $\mathcal{C}$.

Definition 5.7. Let $\mathcal{C}$ be an ∞-category in which colimits and limits indexed over BG exist. Define the homotopy orbits functor

$$\begin{aligned} (-)_{hG} : \mathcal{C}^{BG} &\to \mathcal{C} \\ F &\mapsto \operatorname{colim}_{BG} F \end{aligned}$$

and homotopy fixed points functor

$$\begin{aligned} (-)^{hG} : \mathcal{C}^{BG} &\to \mathcal{C} \\ F &\mapsto \lim_{BG} F \end{aligned}$$

Let $p : BG \to *$ be the canonical projection and $p^* : \mathcal{C} \to \mathcal{C}^{BG}$ be the pullback functor. Then $(-)_{hG}$ is left adjoint to p^* and $(-)^{hG}$ is right adjoint to p^*. Let's put this into a more general context. Let $f : X \to Y$ be a map of Kan complexes, denote $f_!$ and f_* the left adjoint and right adjoint functors of p^*, respectively. We will construct the norm map as a natural transformation $\mathrm{Nm}_f : f_! \to f_*$, So the norm map in our interest becomes a special case.

We still need to impose some conditions on $\mathcal{C}$. The condition we need is to assume $\mathcal{C}$ is a preadditive ∞-category, whose definition directly corresponds to the one in 1-category [HA][Definition 6.1.6.13]. We say that a map of $f : X \to Y$ of Kan complexes is n-truncated if all the homotopy fibers of f has trivial homotopy groups at degree higher than n. Furthermore, we say a 1-truncated map is a *relative finite groupoid* if each fiber of f has finitely many connected components and each of which is a classifying space of a finite group.

We refer the readers to [NS][Construction I.1.7] for the details of the construction of the norm transformation and summarize the result in the following proposition.

Proposition 5.8. *Let $\mathcal{C}$ be a preadditive ∞-category which has limits and colimits over all classifying spaces of finite groups. Let $f : X \to Y$ be a relative finite groupoid of Kan complexes, then both the left adjoint $f_!$ and right adjoint f_* of f^* exist, and there is a natural transformation :*

$$\mathrm{Nm}_f : f_! \to f_*$$

Now we can define the Tate construction in a stable ∞-category $\mathcal{C}$.

Definition 5.9. Let $\mathcal{C}$ be a stable ∞-category which admits all limits and colimits over BG. The Tate construction is the cofiber

$$\begin{aligned} (-)^{tG} : \mathcal{C}^{BG} &\to \mathcal{C} \\ X &\mapsto X^{tG} := \mathrm{cofib}(\mathrm{Nm}_G : X_{hG} \to X^{hG}) \end{aligned}$$

For our interest, we consider the Tate construction in **Sp** the ∞-category of spectra.

Example 5.10. If we take the Eilenberg-Maclane spectrum HM of the G-module M, then we recover the usual Tate cohomology via taking the homotopy groups of the Tate spectrum of HM^{tG}

$$\pi_*(HM^{tG}) \cong \hat{H}^{-*}(G, M)$$

6 tom Dieck Splitting Theorem

The naive fixed points functor $(-)^G$ has some nice properties. One of them is that $(-)^G$ commutes with suspension functor, i.e.

$$\Sigma^\infty A^G \cong (\Sigma^\infty A)^G$$

for any based G-space A. However, the derived fixed points functor F^G does not behave that well with suspension. The tom Dieck splitting theorem measures the difference of commuting suspension and taking derived fixed points spectrum.

Theorem 6.1 (tom Dieck splitting). *Let G be a finite group and A a G-space. Then we have decompositions of the derived fixed points and equivariant stable homotopy groups of the suspension spectrum $\Sigma^\infty A$ of A:*

$$F^G(\Sigma^\infty A) \simeq \bigvee_{(H)\subseteq G} \Sigma^\infty EWH_+ \wedge_{WH} A^H \tag{1}$$

$$\pi_*^G(\Sigma^\infty A) \cong \bigoplus_{(H)\subseteq G} \pi_*^{WH}(\Sigma^\infty EWH_+ \wedge A^H) \tag{2}$$

where the index $(H) \subseteq G$ means the sum is running over all the conjugacy classes of subgroups of G and WH is the Weyl group $WH := N_G H/H$ of H.

As an application of the tom Dieck splitting theorem, let's identify $\pi_0^G(\mathbb{S})$, the 0-th equivariant stable homotopy group of the sphere spectrum. We let $X = S^0$, viewed as a trivial G-space in (2) of the above theorem. Then the left-hand side becomes $\pi_0^G(\mathbb{S})$, the right-side becomes

$$\bigoplus_{(H)\subseteq G} \pi_0^{WH}(\Sigma^\infty EWH_+)$$

a direct sum over the conjugacy classes of subgroups of G.

We first show that each summand $\pi_0^{WH}(\Sigma^\infty EWH_+)$ is isomorphic to $\mathbb{Z}$ additively. Let H be a finite group and EH a contractible CW-complex on which H acts freely. Choose a point $x \in EH$, then it induces an H-equivariant map $a : H \to EH$ by $h \mapsto h \cdot x$. As EH is path-connected, the homotopy type of the map a is independent of the chosen point x and so is the induced map on suspension spectrum $\Sigma^\infty H_+ \to \Sigma^\infty EH_+$.

Lemma 6.2. *The following composite is an isomorphism:*

$$\pi_0(\mathbb{S}) \xrightarrow[\cong]{Tr_e^H} \pi_0^H(\Sigma^\infty H_+) \xrightarrow{\pi_0(a)} \pi_0^H(\Sigma^\infty EH_+)$$

where the first map is the external transfer isomorphism as defined in [Sch][Definition 4.12].

The above lemma tells us that there is an isomorphism of abelian groups

$$\pi_0^G(\mathbb{S}) \cong \bigoplus_{(H)\subseteq G} \mathbb{Z}$$

Moreover, the 0-th equivariant stable homotopy group has a ring structure. Let $f : S^V \to S^V$ and $g : S^W \to S^W$ represent two classes in $\pi_0^G(\mathbb{S})$, then we define their product in $\pi_0^G(\mathbb{S})$ as the class represented by $S^{V\oplus W} \xrightarrow{f\wedge g} S^{V\oplus W}$. We recall the Burnside ring $A(G)$ of a finite group G is the group completion of the isomorphism classes of finite G-sets under direct sum and the multiplication is given by product of G-sets. Note that the Burnside ring $A(G)$ has a $\mathbb{Z}$-basis given by the cosets

$$\big\{[G/H] \mid H \text{ runs over all representatives of conjugacy classes of subgroups of } G\big\}$$

Hence the Burnside ring $A(G)$ is additively isomorphic to the equivariant 0-stem $\pi_0^G(\mathbb{S})$. We define a map $\sigma_G : A(G) \to \pi_0^G(\mathbb{S})$ by sending the basis $[G/H]$ to $tr_H^G(1_H)$, where 1_H represents the identity in $\pi_0^H(\mathbb{S})$.

Theorem 6.3. *[Sch][Theorem 6.14] For every finite group G, the map*

$$\sigma_G : A(G) \to \pi_0^G(\mathbb{S})$$

is a isomorphism of rings. Moreover, the isomorphisms σ_G commute with transfer and restriction along group homomorphisms.

Remark 6.4. The above isomorhpism is actually an isomorphism of Mackey functors whose definitions will be given in Jonathan's talk.

7 Norm Construction

The norm construction is a "multiplicative transfer". Namely, it is the left adjoint functor of the restriction functor from commutative G-ring spectra to commutative H-ring spectra. The main reference for this section is [Sch]. We now introduce the norm construction for orthogonal G-spectra.

Let G be a finite group and H a subgroup of index m. Let $\langle G : H \rangle$ denote the set of m-tuples such that their classes in G/H give a partition of G. That is,

$$\langle G : H \rangle := \{(g_1, \dots, g_m) \in G^m \mid G = \bigcup_{i=1}^{m} g_i H\}$$

Recall the wreath product $\Sigma_m \wr H$ is the semi-direct product $\Sigma_m \ltimes H^m$ with multiplication:

$$(\sigma; h_1, \dots, h_m) \cdot (\tau; k_1, \dots, k_m) = (\sigma\tau; h_{\tau(1)} k_1, \dots, h_{\tau(m)} k_m)$$

The wreath product acts from the right on $\langle G : H \rangle$ by

$$(g_1, \dots, g_m) \cdot (\sigma; h_1, \dots, h_m) = (g_{\sigma(1)} h_1, \dots, g_{\sigma(m)} h_m)$$

Denote $X^{(m)}$ the m-fold symmetric product of an orthogonal H-spectrum X. The symmetric group Σ_m acts on $X^{(m)}$ by permuting factors. The group H acts on each factor and combines to an action of H^m on $X^{(m)}$. Therefore, we have an action of the wreath product $\Sigma_m \wr H$ on $X^{(m)}$. Symbolically, this action reads as

$$(\sigma; h_1, \dots, h_m) \cdot (x_1, \dots, x_m) := (h_{\sigma^{-1}(1)} x_1, \dots, h_{\sigma^{-1}(m)} x_m)$$

We denote the obtained orthogonal $\Sigma_m \wr H$-spectrum by $P^m X$.

Definition 7.1. Let H be a subgroup of G with $[G : H] = m$. The norm of an orthogonal H-spectrum X is the orthogonal G-spectrum defined by

$$N_H^G X := \langle G : H \rangle_+ \wedge_{\Sigma_m \wr H} P^m X$$

The norm $N_H^G X$ has the following properties.

Proposition 7.2. *1. The underlying orthogonal spectrum is isomorphic to $X^{(m)}$.*

2. *The norm functor commutes with smash products up to isomorphism in $\mathbf{Sp_G^O}$, i.e.*

$$N_H^G(X \wedge Y) \cong N_H^G X \wedge N_H^G Y$$

3. *For each H-ring spectrum R, its norm $N_H^G R$ is a G-ring spectrum. Its multiplication map is given by the following composite*

$$N_H^G R \wedge N_H^G R \cong N_H^G(R \wedge R) \xrightarrow{N_H^G \mu} N_H^G R$$

4. *For nested subgroups $K \subseteq H \subseteq G$ and every K-spectrum X, the G-spectra $N_H^G(N_K^H X)$ and $N_K^G X$ are naturally isomorphic.*

5. *The norm functor N_H^G preserves π_*-isomorphisms between cofibrant objects in $\mathbf{Sp}_H^O$, hence it can be left derived to a functor on homotopy category $\mathrm{Ho}(\mathbf{Sp}_H^O) \to \mathrm{Ho}(\mathbf{Sp}_G^O)$.*

6. *For every cofibrant orthogonal H-spectrum X, there is π_*-isomorphism*

$$\Phi^H X \simeq \Phi^G N_H^G X$$

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