Coefficients of equivariant complex cobordism

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Complex cobordism

- Complex manifolds: Compact smooth manifolds, with a tangential stable almost complex structure.
- Two closed manifolds are cobordant, if their disjoint union is the boundary of a third manifold.
- This is an equivalent relation.
- Complex cobordism ring Ω_*^U (graded), under disjoint union and Cartesian product.

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Thom's Theorem

- Thom space $Th(\xi)$.
- Universal n-complex bundle γ^n.
- Thom's homomorphism: $\tau : \pi_{k+2n}Th(\gamma^n) \to \Omega_k^U$.

Theorem (Thom, 54)
τ is an isomorphism for large n.

Those Thom spaces could be assembled to form a spectrum called MU, and $\Omega_*^U \cong \pi_* MU$.

Theorem (Milnor, Novikov, 60)
$MU_* = \pi_* MU = \mathbb{Z}[x_1, x_2, ...]$ where $x_i \in \pi_{2i} MU$.

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Homotopical equivariant complex cobordism MU_G

- Compact Lie group G.
- Complete universe $\mathcal{U}$.
- $B\mathcal{U}(n)$: G-space of n-dimensional complex subspaces of $\mathcal{U}$.
- Universal n-complex G-vector bundle γ_G^n.
- Complex finite dimensional representation V: G-vector bundle over a point.

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Construction (tom Dieck, 70)

For $V \subset W$, there are classifying map $(W - V) \times \gamma_G^{|V|} \to \gamma_G^{|W|}$.

We have

$$Th((W - V) \times \gamma_G^{|V|}) \cong \Sigma^{W-V} Th(\gamma_G^{|V|}) \to Th(\gamma_G^{|W|}).$$

Let $D_V = Th(\gamma_G^{|V|})$ with the structured maps described above, then spectrify to obtain MU_G.

MU_G is a genuine multiplicative G-specturm. It is complex stable:

$$MU_G^*(X) \cong MU_G^{*+2|V|}(S^V \wedge X).$$

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Geometric equivariant complex cobordism

- Tangential stable almost complex structure for a smooth G-manifold M: equivariant isomorphism to a G-complex vector bundle ξ over M:

$$TM \times \mathbb{R}^k \cong \xi.$$

- Geometric equivariant complex cobordism ring Ω_*^G.

However,

$$\Omega_*^G \ncong \pi_* MU_G.$$

The Euler class $e_V \in \pi_{-2|V|} MU_G$ of V is

$$S^0 \to S^V \to Th(\gamma_G^{|V|}).$$

Fact: $e_V \neq 0$ if $V^G = 0$.

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Pontryagin-Thom construction

- Take a cobordant class $[M]$.
- Equivariant Whiteny's embedding: $M \hookrightarrow V$.
- The normal bundle ν embeds as a tubular neighborhood.

Pontryagin-Thom constuction gives a composite map

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The opposite of Thom's homomorphism does not exist, due to transversality issues.

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History

We expect MU_G to play the same key role as MU plays in non-equivariant homotopy theory. MU is the universal complex oriented cohomology theory, and its coefficient ring MU_* admits a universal formal group law.

- $G = \mathbb{Z}/p$: Greenlees, May, Kosniowski, Kriz, Strickland, ...
- $G = S^1, T$: Sinha.
- G finite abelian: Abram, Kriz.
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Theorem (Comezaña, 96)
If G is abelian, then $\pi_* MU_G$ is a free MU_*-module concentrated in even degrees.

In 1997, Greenlees and May proved a localization and completion theorem for MU_G-module spectra.

Theorem. For G abelian, $(MU_G^*)_J^\wedge \cong MU^*(BG)$, here J is the kernel of the augmentation map $(MU_G)_* \to MU_*$.

The augmentation ideal J contains all Euler classes e_V.

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Tate diagram

Consider cofiber sequence of $\mathbb{Z}/p$-spaces

$$E\mathbb{Z}/p_+ \to S^0 \to \widetilde{E\mathbb{Z}/p}.$$

The Tate diagram for $MU_{\mathbb{Z}/p}$:

$$\begin{array}{ccccc}
E\mathbb{Z}/p_+ \wedge MU & \longrightarrow & MU & \longrightarrow & \widetilde{E\mathbb{Z}/p} \wedge MU \\
\downarrow \sim & & \downarrow & & \downarrow \\
E\mathbb{Z}/p_+ \wedge F(E\mathbb{Z}/p_+, MU) & \longrightarrow & F(E\mathbb{Z}/p_+, MU) & \longrightarrow & \widetilde{E\mathbb{Z}/p} \wedge F(E\mathbb{Z}/p_+, MU)
\end{array}$$

A closer look

Take fixed points $(-)^{\mathbb{Z}/p}$:

$$\begin{array}{ccc} (MU_{\mathbb{Z}/p})^{\mathbb{Z}/p} & \longrightarrow & \Phi^{\mathbb{Z}/p} MU_{\mathbb{Z}/p} \\ \downarrow & & \downarrow \\ (F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p}))^{\mathbb{Z}/p} & \longrightarrow & (t(MU_{\mathbb{Z}/p}))^{\mathbb{Z}/p} \end{array}$$

- Tom Dieck computes the geometric fixed point $\Phi^{\mathbb{Z}/p} MU_{\mathbb{Z}/p}$.
- The coefficient of the bottom left is $MU^*(B\mathbb{Z}/p)$.
- Let F be the universal fgl, $MU^*(B\mathbb{Z}/p) = MU_*[[u]]/([p]_F u)$.
- The bottom map is localization at u (Greenlees, May).

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A pullback square

Theorem (Kriz, 99)
There is a pullback square of rings:

$$\begin{array}{ccc}
(MU_{\mathbb{Z}/p})_* & \longrightarrow & MU_*[b_i^k, (b_0^k)^{-1} \mid i \geq 0, k \in (\mathbb{Z}/p)^\times] \\
\downarrow & & \downarrow \phi \\
MU_*[[u]]/([p]_F u) & \longrightarrow & MU_*[[u]]/([p]_F u)[u^{-1}]
\end{array}$$

Here $|b_i^k| = 2i - 2$, and ϕ sends b_i^k to the coefficient of x^i in $x +_F [k]_F u$. In particular, $\phi(b_0^k) = [k]_F u$.

Generators and relations

Strickland first gives an explicit structure for $MU^*_{\mathbb{Z}/2}$.

Theorem (Strickland, 01) Let the universal formal group law be $F(x, y) = \sum a_{i,j} x^i y^j$.
$MU^*_{\mathbb{Z}/2}$ is generated over MU^* by elements $u, b_{i,j}, q_i$ for $i, j \geq 0$ subject to the following relations:

- $b_{0,0} = u, b_{0,1} = 1, b_{0,\geq 2} = 0,$
- $b_{i,j} - a_{i,j} = ub_{i,j+1},$
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i.e., $MU^*_{\mathbb{Z}/2} = MU^*[u, b_{i,j}, q_i \mid i, j \geq 0]/\sim.$

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Generalization

This method generalizes Strickland's result to other $\mathbb{Z}/p$, even to $\mathbb{Z}/(p^n)$.

Theorem. $MU^*_{\mathbb{Z}/p}$ is generated over MU^* by elements $u, b^k_{i,j}, (b^k_{0,1})^{-1}, q_i$ for $i \geq 0, j \geq 1, k \in (\mathbb{Z}/p)^\times$ with relations

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Main result - notations

- $G = \Sigma_3$,
- α is the sign representation of Σ_3,
- γ is the standard representation of Σ_3.

Tate diagram for families, which gives us building blocks:

$$\begin{array}{ccc} MU_{\Sigma_3} & \longrightarrow & S^{\infty\alpha} \wedge MU_{\Sigma_3} \\ \downarrow & & \downarrow \\ F(S(\infty\alpha)_+, MU_{\Sigma_3}) & \longrightarrow & S^{\infty\alpha} \wedge F(S(\infty\alpha)_+, MU_{\Sigma_3}) \end{array}$$

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Main result

Theorem (Hu, Kriz, L.) The ring $(MU_{\Sigma_3})_*$ is the limit of the diagram of rings:

$$
\begin{array}{ccccc}
 & & & & R \\
 & & & & \downarrow \\
 & & ((MU_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2} & \longrightarrow & MU_*[(u_\gamma)^{\pm 1}, b_{2i}^{\gamma}]/2 \\
 & & \downarrow{\scriptstyle res} & & \\
(MU_{\mathbb{Z}/2})_* & \xrightarrow{res} & MU_* & &
\end{array}
$$

Outline of computation

- Calculate $MU^* B\Sigma_3 = MU_*[[u_\alpha, u_\gamma]]/([2]u_\alpha, \{3\}u_\gamma)$,
- Calculate $(S^{\infty\alpha} \wedge MU_{\Sigma_3})_*$ in the pullback diagram for $\mathcal{F}[\Sigma_3]$,

$$\begin{array}{ccc}
S^{\infty\alpha} \wedge MU_{\Sigma_3} & \longrightarrow & \widetilde{E\mathcal{F}[\Sigma_3]} \wedge MU_{\Sigma_3} \\
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S^{\infty\alpha} \wedge F(S(\infty\gamma)_+, MU_{\Sigma_3}) & \longrightarrow & \widetilde{E\mathcal{F}[\Sigma_3]} \wedge F(S(\infty\gamma)_+, MU_{\Sigma_3})
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$(S^{\infty\alpha} \wedge MU_{\Sigma_3})_*$ is product of $(\Phi^{\Sigma_3} MU_{\Sigma_3})_*$ and $(\Phi^{\mathbb{Z}/2} MU_{\mathbb{Z}/2})_*$.

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$$\begin{array}{ccc} S^{\infty\alpha} \wedge MU_{\Sigma_3} & \longrightarrow & \widetilde{E\mathcal{F}[\Sigma_3]} \wedge MU_{\Sigma_3} \\ \downarrow & & \downarrow \\ S^{\infty\alpha} \wedge F(S(\infty\gamma)_+, MU_{\Sigma_3}) & \longrightarrow & \widetilde{E\mathcal{F}[\Sigma_3]} \wedge F(S(\infty\gamma)_+, MU_{\Sigma_3}) \end{array}$$

$(S^{\infty\alpha} \wedge MU_{\Sigma_3})_*$ is product of $(\Phi^{\Sigma_3} MU_{\Sigma_3})_*$ and $(\Phi^{\mathbb{Z}/2} MU_{\mathbb{Z}/2})_*$.

Outline of computation

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Outline of computation (continued)

Calculate $(F(S(\infty\alpha)_+, MU_{\Sigma_3})_*$, it is the limit of the diagram of rings (glueing pullback diagrams):

$$
\begin{array}{ccccc}
 & & & & MU_*[(u_\gamma)^{\pm 1}, b^\gamma_{2i}][[u_\alpha]]/[2]u_\alpha \\
 & & & & \Big\downarrow{\scriptstyle u_\alpha \mapsto 0} \\
 & & ((MU_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2} & \longrightarrow & MU_*[(u_\gamma)^{\pm 1}, b^\gamma_{2i}]/2 \\
 & & \Big\downarrow{\scriptstyle res} & & \\
MU^* B\mathbb{Z}/2 & \longrightarrow & MU_* & &
\end{array}
$$

Outline of computation (continued)

$$\begin{array}{ccc} MU_{\Sigma_3} & \longrightarrow & S^{\infty\alpha} \wedge MU_{\Sigma_3} \\ \downarrow & & \downarrow \\ F(S(\infty\alpha)_+, MU_{\Sigma_3}) & \longrightarrow & S^{\infty\alpha} \wedge F(S(\infty\alpha)_+, MU_{\Sigma_3}) \end{array}$$

The bottom map is inversion of u_α.

Put it altogether, R is the pullback of the following diagram:

$$\begin{array}{ccc} & & (\Phi^{\Sigma_3} MU_{\Sigma_3})_* \\ & & \downarrow \\ MU_*[(u_\gamma)^{\pm 1}, b_{2i}^{\gamma}][[u_\alpha]]/[2]u_\alpha & \longrightarrow & u_\alpha^{-1} MU_*[(u_\gamma)^{\pm 1}, b_{2i}^{\gamma}][[u_\alpha]]/[2]u_\alpha \end{array}$$

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Main result

Theorem (Hu, Kriz, L.) The ring $(MU_{\Sigma_3})_*$ is the limit of the diagram of rings:

$$\begin{array}{ccccc}
 & & & & R \\
 & & & & \downarrow \\
 & & ((MU_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2} & \longrightarrow & MU_*[(u_\gamma)^{\pm 1}, b_{2i}^{\gamma}]/2 \\
 & & \downarrow \scriptstyle{res} & & \\
(MU_{\mathbb{Z}/2})_* & \xrightarrow{res} & MU_* & &
\end{array}$$

Equivariant formal group laws

Non-equivariantly, $k[[y]] \to k[[y \otimes 1, 1 \otimes y]]$.

For finite abelian group A: (Cole, Greenlees, Kriz, 00),

- A commutative topological Hopf k-algebra (R, Δ), complete at ideal I,
- A map $\theta : R \to k^{A^*}$ ($A^* = \mathrm{Hom}(A, S^1)$) of Hopf k-algebras, and $I = ker(\theta)$,
- A regular element $y(\epsilon) \in R$ that generates $ker(\theta_\epsilon)$, and $R/ker(\theta_\epsilon) \cong k$.

There exists universal ring L_A, such that A-fgl$(k) \cong$ Ring(L_A, k).

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Complex oriented equivariant cohomology theories

An orientation class $x \in E_A^*(\mathbb{C}P(\mathcal{U}), pt)$.

Theorem (Cole, 96)
Given a complete flag $V^0 \subset V^1 \subset ...$ as a filtration of $\mathcal{U}$:

$$E_A^*(\mathbb{C}P(\mathcal{U})) = E_A^*\{\{y(V^0) = 1, y(V^1), y(V^2), ...\}\}.$$

A complex oriented cohomology theory E_A^* gives rise to an A-equivariant formal group law:

- $k = E_A^*$, $R = E_A^*(\mathbb{C}P(\mathcal{U}))$,
- Δ is induced by $\mathbb{C}P(\mathcal{U}) \times \mathbb{C}P(\mathcal{U}) \to \mathbb{C}P(\mathcal{U})$,
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Equivariant Quillen's Theorem

Theorem (Quillen, 69)
The canonical map $L \to MU^*$ is an isomorphism.

Theorem (Greenlees, 01)
The canonical map $\lambda_A : L_A \to MU_A^*$, is surjective, and its kernel is Euler torsion and infinitely Euler divisible.

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Thank you for listening!