Coefficients of equivariant complex cobordism

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- Complex manifolds: Compact smooth manifolds, with a tangential stable almost complex structure.
- Two closed manifolds are cobordant, if their disjoint union is the boundary of a third manifold.
- This is an equivalent relation.
- Complex cobordism ring Ω^U_* (graded), under disjoint union and Cartesian product.

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Thom's Theorem

• Thom space $Th(\xi)$.

- Universal *n*-complex bundle γ^n .
- Thom's homomorphism: $\tau : \pi_{k+2n} Th(\gamma^n) \to \Omega_k^U$.

Theorem (Thom, 54)

au is an isomorphism for large n.

Those Thom spaces could be assembled to form a spectrum called MU, and $\Omega^U_* \cong \pi_* MU$.

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- Compact Lie group G.
- Complete universe U.
- BU(n): G-space of *n*-dimensional complex subspaces of U.
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Construction (tom Dieck, 70)

For $V \subset W$, there are classifying map $(W - V) \times \gamma_G^{|V|} \to \gamma_G^{|W|}$. We have

$$Th((W - V) \times \gamma_G^{|V|}) \cong \Sigma^{W-V} Th(\gamma_G^{|V|}) \to Th(\gamma_G^{|W|}).$$

Let $D_V = Th(\gamma_G^{|V|})$ with the structured maps described above, then spectrify to obtain MU_G .

 MU_G is a genuine multiplicative *G*-specturm. It is complex stable:

$$MU_G^*(X) \cong MU_G^{*+2|V|}(S^V \wedge X).$$

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Geometric equivariant complex cobordism

 Tangential stable almost complex structure for a smooth G-manifold M: equivariant isomorphism to a G-complex vector bundle ξ over M:

$$TM \times \mathbb{R}^k \cong \xi.$$

• Geometric equivariant complex cobordism ring Ω^{G}_{*} .

However,

$$\Omega^G_* \cong \pi_* M U_G.$$

The Euler class $e_V \in \pi_{-2|V|} M U_G$ of V is

$$S^0 \to S^V \to Th(\gamma_G^{|V|}).$$

Fact: $e_V \neq 0$ if $V^G = 0$.

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• Take a cobordant class [*M*].

- Equivariant Whiteny's embedding: $M \hookrightarrow V$.
- The normal bundle ν embeds as a tubular neighborhood.

Pontryagin-Thom constuction gives a composite map

$$S^{V} \to Th(\nu) \to Th(\gamma_{G}^{|\nu|}),$$

which induces a homomorphism $\Omega^G_* \to \pi_* MU_G$.

The opposite of Thom's homomorphism does not exist, due to transversality issues.

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We expect MU_G to play the same key role as MU plays in non-equivariant homotopy theory. MU is the universal complex oriented cohomology theory, and its coefficient ring MU_* admits a universal formal group law.

- $G = \mathbb{Z}/p$: Greenlees, May, Kosniowski, Kriz, Strickland, ...
- $G = S^1, T$: Sinha.
- *G* finite abelian: Abram, Kriz.
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If *G* is abelian, then π_*MU_G is a free MU_* -module concentrated in even degrees.

In 1997, Greenlees and May proved a localization and completion theorem for MU_G -module spectra.

Theorem. For *G* abelian, $(MU_G^*)_J^{\wedge} \cong MU^*(BG)$, here *J* is the kernel of the augmentation map $(MU_G)_* \to MU_*$.

The augmentation ideal J contains all Euler classes e_V .

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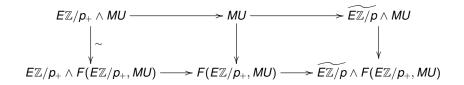
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Tate diagram

Consider cofiber sequence of \mathbb{Z}/p -spaces

$$E\mathbb{Z}/p_+ \to S^0 \to \widetilde{E\mathbb{Z}/p}.$$

The Tate diagram for $MU_{\mathbb{Z}/p}$:



A closer look

Take fixed points $(-)^{\mathbb{Z}/p}$:

- Tom Dieck computes the geometric fixed point $\Phi^{\mathbb{Z}/p}MU_{\mathbb{Z}/p}$.
- The coefficient of the bottom left is $MU^*(B\mathbb{Z}/p)$.
- Let *F* be the universal fgl, $MU^*(B\mathbb{Z}/p) = MU_*[[u]]/([p]_F u)$.
- The bottom map is localization at *u* (Greenlees, May).

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Theorem (Kriz, 99)

There is a pullback square of rings:

Here $|b_i^k| = 2i - 2$, and ϕ sends b_i^k to the coefficient of x^i in $x +_F [k]_F u$. In particular, $\phi(b_0^k) = [k]_F u$.

Strickland first gives an explicit structure for $MU^*_{\mathbb{Z}/2}$.

Theorem (Strickland, 01) Let the universal formal group law be $F(x, y) = \sum a_{i,j} x^i y^j$.

 $MU_{\mathbb{Z}/2}^*$ is generated over MU^* by elements $u, b_{i,j}, q_i$ for $i, j \ge 0$ subject to the following relations:

•
$$b_{0,0} = u, b_{0,1} = 1, b_{0,\geq 2} = 0,$$

•
$$b_{i,j} - a_{i,j} = ub_{i,j+1}$$
,

•
$$q_0 = 0, q_i - b_{i,0} = uq_{i+1}$$
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i.e., $MU^*_{\mathbb{Z}/2} = MU^*[u, b_{i,j}, q_i \,|\, i, j \ge 0]/\sim$.

The method is to combine the pullback square with localization and completion theorems.

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Generalization

This method generalizes Strickland's result to other \mathbb{Z}/p , even to $\mathbb{Z}/(p^n)$.

Theorem. $MU_{\mathbb{Z}/p}^*$ is generated over MU^* by elements $u, b_{i,j}^k, (b_{0,1}^k)^{-1}, q_i$ for $i \ge 0, j \ge 1, k \in (\mathbb{Z}/p)^{\times}$ with relations

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Here $a_{i,j}^k$ is the coefficient of $x^i u^j$ in $x +_F [k]u$, and c_i is the coefficient of u^i in [p]u.

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Theorem. $MU_{\mathbb{Z}/p}^*$ is generated over MU^* by elements $u, b_{i,j}^k, (b_{0,1}^k)^{-1}, q_i \text{ for } i \ge 0, j \ge 1, k \in (\mathbb{Z}/p)^{\times}$ with relations • $b_{0,1}^1 = 1, b_{0,\ge 2}^1 = 0,$ • $b_{i,j}^k - a_{i,j}^k = ub_{i,j+1}^k,$ • $q_0 = 0, q_i - c_i = uq_{i+1}.$

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• α is the sign representation of Σ_3 ,

• γ is the standard representation of Σ_3 .

Tate diagram for families, which gives us building blocks:



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Tate diagram for families, which gives us building blocks:

$$\begin{array}{c} \mathcal{MU}_{\Sigma_{3}} \longrightarrow S^{\infty\alpha} \wedge \mathcal{MU}_{\Sigma_{3}} \\ \downarrow \\ \mathcal{F}(S(\infty\alpha)_{+}, \mathcal{MU}_{\Sigma_{3}}) \longrightarrow S^{\infty\alpha} \wedge \mathcal{F}(S(\infty\alpha)_{+}, \mathcal{MU}_{\Sigma_{3}}) \end{array}$$

Main result

Theorem (Hu, Kriz, L.) The ring $(MU_{\Sigma_3})_*$ is the limit of the diagram of rings:

$$((MU_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2} \longrightarrow MU_*[(u_{\gamma})^{\pm 1}, b_{2i}^{\gamma}]/2$$

$$\downarrow^{res}_{q}$$

$$(MU_{\mathbb{Z}/2})_* \xrightarrow{res} MU_*$$

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Outline of computation

• Calculate $MU^*B\Sigma_3 = MU_*[[u_\alpha, u_\gamma]]/([2]u_\alpha, \{3\}u_\gamma),$

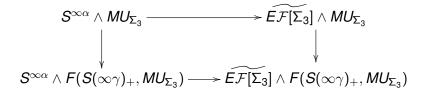
Calculate (S^{∞α} ∧ MU_{Σ₃})_{*} in the pullback diagram for F[Σ₃],



 $(S^{\infty lpha} \wedge MU_{\Sigma_3})_*$ is product of $(\Phi^{\Sigma_3}MU_{\Sigma_3})_*$ and $(\Phi^{\mathbb{Z}/2}MU_{\mathbb{Z}/2})_*$.

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- Calculate $MU^*B\Sigma_3 = MU_*[[u_\alpha, u_\gamma]]/([2]u_\alpha, \{3\}u_\gamma),$
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 $(S^{\infty \alpha} \wedge MU_{\Sigma_3})_*$ is product of $(\Phi^{\Sigma_3}MU_{\Sigma_3})_*$ and $(\Phi^{\mathbb{Z}/2}MU_{\mathbb{Z}/2})_*$.

Outline of computation (continued)

Calculate $(F(S(\infty \alpha)_+, MU_{\Sigma_3})_*)$, it is the limit of the diagram of rings (glueing pullback diagrams):

$$MU_*[(u_{\gamma})^{\pm 1}, b_{2i}^{\gamma}][[u_{\alpha}]]/[2]u_{\alpha}$$

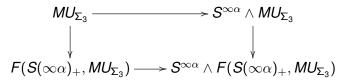
$$\downarrow u_{\alpha \mapsto 0}$$

$$((MU_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2} \longrightarrow MU_*[(u_{\gamma})^{\pm 1}, b_{2i}^{\gamma}]/2$$

$$\downarrow res$$

$$MU^*B\mathbb{Z}/2 \longrightarrow MU_*$$

Outline of computation (continued)



The bottom map is inversion of u_{α} . Put it altogether, *R* is the pullback of the following diagram:

 $MU_{*}[(u_{\gamma})^{\pm 1}, b_{2i}^{\gamma}][[u_{\alpha}]]/[2]u_{\alpha} \longrightarrow u_{\alpha}^{-1}MU_{*}[(u_{\gamma})^{\pm 1}, b_{2i}^{\gamma}][[u_{\alpha}]]/[2]u_{\alpha}$

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Outline of computation (continued)

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Main result

Theorem (Hu, Kriz, L.) The ring $(MU_{\Sigma_3})_*$ is the limit of the diagram of rings:

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$$(MU_{\mathbb{Z}/2})_* \xrightarrow{res} MU_*$$

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For finite abelian group A: (Cole, Greenlees, Kriz, 00),

- A commutative topological Hopf k-algebra (R, △), complete at ideal I,
- A map θ : $R \rightarrow k^{A^*}$ ($A^* = \text{Hom}(A, S^1)$) of Hopf *k*-algebras, and $l = ker(\theta)$,
- A regular element $y(\epsilon) \in R$ that generates $ker(\theta_{\epsilon})$, and $R/ker(\theta_{\epsilon}) \cong k$.

There exists universal ring L_A , such that A-fgl(k) \cong Ring(L_A , k).

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Equivariant formal group laws

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Complex oriented equivariant cohomology theories

An orientation class $x \in E^*_A(\mathbb{C}P(\mathcal{U}), pt)$.

Theorem (Cole, 96) Given a complete flag $V^0 \subset V^1 \subset ...$ as a filtration of \mathcal{U} :

$$E_A^*(\mathbb{C}P(\mathcal{U})) = E_A^*\{\{y(V^0) = 1, y(V^1), y(V^2), ...\}\}.$$

A complex oriented cohomology theory E_A^* gives rise to an *A*-equivariant formal group law:

• $k = E_A^*, R = E_A^*(\mathbb{C}P(\mathcal{U})),$

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Equivariant Quillen's Theorem

Theorem (Quillen, 69) The canonical map $L \rightarrow MU^*$ is an isomorphism.

Theorem (Greenlees, 01) The canonical map $\lambda_A : L_A \to MU_A^*$, is surjective, and its kernel is Euler torsion and infinitely Euler divisible.

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Thank you for listening!

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