Converse of Smith Theory

Min Yan
Hong Kong University of Science and Technology

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Joint with S. Cappell and S. Weinberger

- Smith Theory and Pseudo-equivalence
- General Action: Oliver
- Semi-Free Action: Jones

Always assume: Finite CW-complex, Finite group

1. Smith Theory

Paul Althaus Smith
Fixed-Point Theorems for Periodic Transformations.
Amer. J. Math., 63(1):1-8, 1941.

THEOREM I(α). *Let p be a prime and M a finite dimensional locally bicompact space which is acyclic mod p. Every homeomorphic transformation of period p^α ($\alpha > 0$) of M into itself admits at least one fixed point.*

THEOREM II. *The totality L of fixed points which theorem $I(1)$ asserts to be non-empty, is acyclic modulo p.*

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THEOREM II. *The totality L of fixed points which theorem $I(1)$ asserts to be non-empty, is acyclic modulo p.*

Theorem
$G = \mathbb{Z}_{p^k}$ acts on $\mathbb{F}_p$-acyclic $X \implies X^G$ is $\mathbb{F}_p$-acyclic.

$$\tilde{H}_*(X; \mathbb{F}_p) = 0 \implies \tilde{H}_*(X^G; \mathbb{F}_p) = 0.$$

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3. Any G, semi-free action ($G_x = G$ or e), and p dividing $|G|$.

Need to divide into two cases:

- Semi-free action: Smith condition must be satisfied.
- General action, $|G|$ is not prime power: Smith condition needs not be satisfied.

The first was studied by Lowell Jones. The second was studied by Robert Oliver.

1. Converse of Smith

Theorem [Lowell Jones 1971]
F is $\mathbb{Z}_n$-acyclic
$\implies F = X^{\mathbb{Z}_n}$ for a contractible X with semi-free $\mathbb{Z}_n$-action.

Remark $\mathbb{Z}_n$-acyclic $\iff \mathbb{Z}_p$-acyclic for all $p|n$.

Theorem [Robert Oliver 1975]
For any G such that $|G|$ is not prime power, there is n_G, such that
$F = X^G$ for a contractible X with G-action
$\iff \chi(F) = 1 \bmod n_G$.

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Definition A G-map is a pseudo-equivalence if it is a homotopy equivalence after forgetting the G-action.

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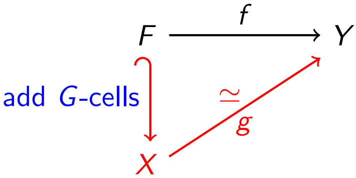

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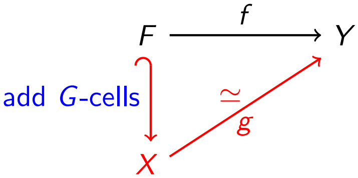

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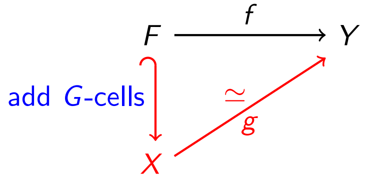

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Jones and Oliver: The case Y is a single point.
Our problem: Y not contractible, especially $\pi = \pi_1 Y$ non-trivial.

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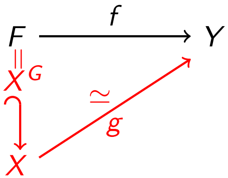

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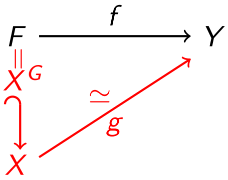

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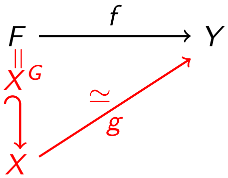

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Pseudo-equivalence: $\pi_1 X \cong \pi_1 Y$ and $H_*(X;\mathbb{Z}\pi) \cong H_*(Y;\mathbb{Z}\pi)$

2. General Action: Pseudo-equivalence Invariant

$|G|$ is not prime power
$\Longrightarrow$ There is n_G, $\chi(X^G) = 1 \bmod n_G$ for contractible G-space X.
$\Longrightarrow$ For pseudo-equiv $g\colon X \to Y$, $\chi(X^G) = \chi(Y^G) \bmod n_G$.
$\Longrightarrow$ $\chi(X^G) \bmod n_G$ is pseudo-equivalence invariant.

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Remark Pseudo-equivalence has no inverse. To get equivalence relation, need zig-zaging sequence of pseudo-equivalences

$$X \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \cdots \bullet \leftarrow \bullet \rightarrow Y$$

$\chi(X^G) \bmod n_G$ is an invariant in this sense.

2. General Action: Main Theorem

Theorem
Suppose $|G|$ is not prime power, and $Y_1^G, Y_2^G, \ldots, Y_k^G$ are components of Y^G. Then there is a subgroup $N_Y \subset \mathbb{Z}^k$, such that $f\colon F \to Y$ can be extended to a pseudo-equivalence G-map $g\colon X \to Y$, with $X^G = F$, if and only if

$$(\ \chi(F_1) - \chi(Y_1^G),\ \ldots,\ \chi(F_k) - \chi(Y_k^G)\ ) \in N_Y, \quad F_i = f^{-1}(Y_i^G).$$

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$$n_G\mathbb{Z}^k \subset N_Y \subset \{(a_i)\colon\ n_G \text{ divides } \sum a_i\}.$$

First $\subset$: component-wise $\chi(F_i) = \chi(Y_i^G)$ mod n_G is sufficient.
Second $\subset$: global $\chi(F) = \chi(Y^G)$ mod n_G is necessary.

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$N_Y = n_G\mathbb{Z}$ for $k = 1$, i.e., Y^G is connected.

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Corollary
Suppose $|G|$ is not prime power, and Y^G is non-empty and connected. Then $F = X^G$ for some X pseudo-equivalent to Y (no direct map needed) if and only if $\chi(F) = \chi(Y^G)$ mod n_G.

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Corollary

If $|G|$ is not prime power, Y is connected, $\chi(Y) = 0 \bmod n_G$, then G acts on a homotopy Y with no fixed points.

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If the action has fixed point, then the homomorphism lifts to $G \to \mathrm{Aut}(\pi)$.

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Theorem

Suppose $|G|$ is not prime power. Then there is an aspherical manifold M with centerless fundamental group, such that $G \to \mathrm{Out}(\pi)$ lifts to $\mathrm{Aut}(\pi)$, and the action has no fixed point.

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Need to show

1. $N_Y = \{(\chi(X_i^G) - \chi(Y_i^G))_{i=1}^k : \text{pseudo-equiv } X \to Y\}$ is an abelian subgroup.
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- Oliver's argument is relative, allowing induction on cells.
- Treat $F = \emptyset$ by the special case $Y = S^1$.

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Corollary If $|G|$ is not prime power, then there is a G-space $X \simeq S^1$, such that $X^G = \emptyset$.

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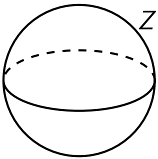

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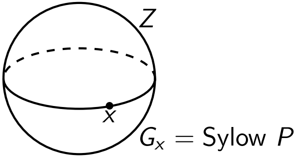

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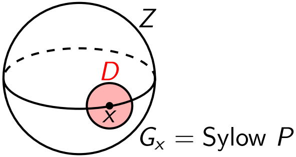

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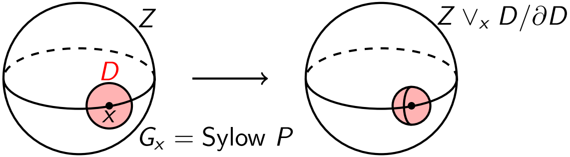

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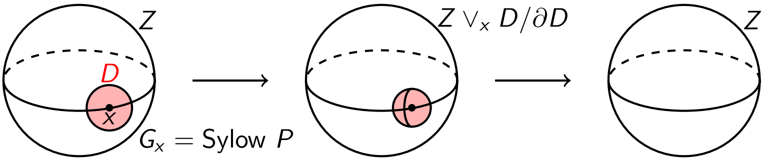

We have $D/\partial D \cong_P Z$ and G-map of degree $1 \pm |G/N_P|$

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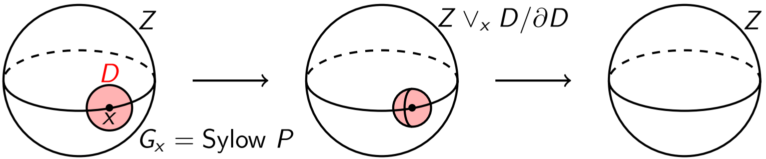

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Repeat and modify for all P_i, get G-map $\phi \colon Z \to Z$ of degree

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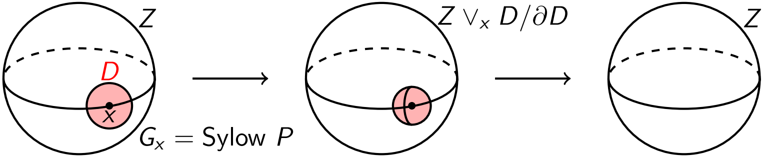

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Mapping torus $X = T(\phi) \to S^1$ is $\simeq$.

2. General Action: Local vs Global Euler

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$N_Y = \mathbb{Z}^k$ (no Euler condition) if $n_G = 1$. By Oliver (1975), this means G is not of the form

$$P \lhd H \lhd G, \quad |P| \text{ and } |G/H| \text{ prime power, and } H/P \text{ cyclic.}$$

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$$P \lhd H \lhd G, \quad |P| \text{ and } |G/H| \text{ prime power, and } H/P \text{ cyclic.}$$

Theorem
Suppose $n_G = 0$, which means

$$P \lhd G, \quad |P| \text{ prime power, and } G/P \text{ cyclic.}$$

Let Γ on $\tilde{Y}$ be the lifting of G on Y. If the connected components $Y_1^G, \ldots, Y_k^G$ of Y^G satisfy

1. Induced splittings $G \xrightarrow{s_i} \Gamma$ are not π-conjugate.
2. $\pi_1 Y_i^G \to \pi_1 Y$ are injective.

Then $N_Y = n_G\mathbb{Z}^k$.

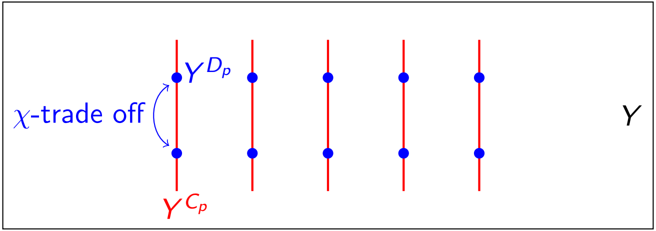
2. General Action: Local vs Global Euler

Theorem [Oliver and Petrie 1982]

Consider $G = D_p$, the dihedral group of order $2p$ (p an odd prime). Consider $f : F \to Y$, Y simply connected.

Let $Y_1^{C_p}, \ldots, Y_l^{C_p}$ be connected components of Y^{C_p}. Then f has pseudo-equivalence extension if and only if

$$\chi(F \cap f^{-1}(Y_i^{C_p})) = \chi(Y^G \cap Y_i^{C_p}) \text{ for all } i.$$



3. Semi-free Action: Pseudo-equivalence Invariant

For semi-free action, pseudo-equivalence $g\colon X \to Y$ implies Smith condition

$$H_*(X^G; \mathbb{F}_p\pi) \cong H_*(Y^G; \mathbb{F}_p\pi), \quad p | \, |G|.$$

So $H_*(-^G; \mathbb{F}_p\pi)$ is pseudo-equivalence invariant, not as easy to use as Euler number.

3. Semi-free Action: Main Theorem

Theorem (fixed target)

A map $f\colon F \to Y$ (no G-action) has pseudo-equivalent extension $g\colon X \to Y$, with semi-free G-space X and $F = X^G$, if and only if

1. *Smith:* $H_*(F; \mathbb{F}_p\pi) \cong H_*(Y; \mathbb{F}_p\pi)$,
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Remark $C(\tilde{f})$ is $\mathbb{Z}\pi$-chain complex, regarded as $\mathbb{Z}[\pi \times G]$-chain complex by trivial G-action. Then Smith condition implies $C(\tilde{f})$ has finite $\mathbb{Z}[\pi \times G]$-projective resolution.

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Theorem (semi-free target)

Consider G acting semi-freely on Y and $f\colon F \to Y^G \subset Y$, exists exists if and only if Smith condition is satisfied and K-theory obstruction $[C(\tilde{f})] \in \tilde{K}_0(\mathbb{Z}[\Gamma])$ *vanishes.*

3. Semi-free Action: Proof

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Attach free G-cells to F to get isomorphism on π_1 and then inductively kill $H_i(f; \mathbb{Z}\pi)$. Get $f^n \colon X^n \to Y$, $n > \dim F, \dim Y$, such that $H_i(f^n; \mathbb{Z}\pi) = 0$ for $i \leq n$. Get exact sequence

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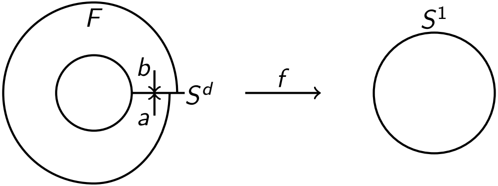
The obstruction is the "stable $\mathbb{Z}(\pi \times G)$-freeness" of $H_{n+1}(f^n;\mathbb{Z}\pi)$. Since $C_*(\tilde{X}^n, \tilde{F})$ is $\mathbb{Z}(\pi \times G)$-free, the obstruction is

$$\pm[H_{n+1}(f^n;\mathbb{Z}\pi)] = [C_*(\tilde{f}^n)] = [C_*(\tilde{f})] \in \tilde{K}_0(\mathbb{Z}[\pi \times G]).$$

3. Semi-Free Action: Example

$Y = S^1$, $\pi = \{t^i : i \in \mathbb{Z}\}$. F is double mapping torus $T(a, b)$ of maps $S^d \to S^d$ of deg a, b



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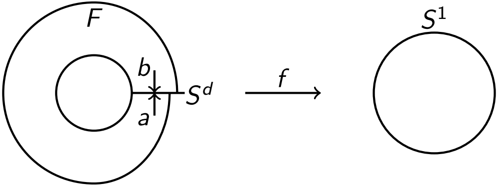

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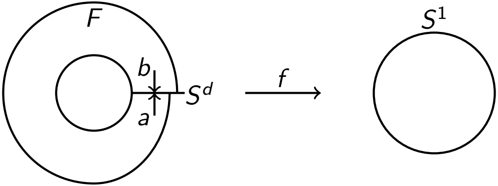

For $G = \mathbb{Z}_n$, we want to extend $f\colon F = T(a,b) \to Y$ to semi-free pseudo-equivalence.
The only non-trivial $\mathbb{Z}\pi$-homology of f is

$$H = H_d(f; \mathbb{Z}[t, t^{-1}]) = \mathbb{Z}[t, t^{-1}]/(at - b).$$

3. Semi-Free Action: Example

The pullback diagrams

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}_n][t,t^{-1}] & \longrightarrow & \frac{\mathbb{Z}[\mathbb{Z}_n]}{\Sigma}[t,t^{-1}] \\ \downarrow & & \downarrow \\ \mathbb{Z}[t,t^{-1}] & \longrightarrow & \mathbb{Z}_n[t,t^{-1}] \end{array} \qquad \begin{array}{ccc} \mathbb{Z}[\mathbb{Z}_n] & \longrightarrow & \frac{\mathbb{Z}[\mathbb{Z}_n]}{\Sigma} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}_n \end{array}$$

induce ∂ between Bass-Heller-Swan decompositions

$$\begin{array}{l} K_1(\mathbb{Z}_n[t,t^{-1}]) = K_1(\mathbb{Z}_n) \oplus K_0(\mathbb{Z}_n) \oplus NK_1(\mathbb{Z}_n) \oplus NK_1(\mathbb{Z}_n) \\ \qquad\qquad \downarrow \partial \\ \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_n][t,t^{-1}]) = \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_n]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}_n]) \oplus NK_0(\mathbb{Z}[\mathbb{Z}_n]) \oplus NK_0(\mathbb{Z}[\mathbb{Z}_n]) \end{array}$$

3. Semi-Free Action: Example 1

For $G = \mathbb{Z}_n$, $n = p^k$, p prime, $a = p$, $b = 1$, we have $H = \mathbb{Z}[t, t^{-1}]/(pt - 1)$, Smith condition satisfied.

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- For $k = 1$, $[(p - 1)^{-1}p] \in NK_1(\mathbb{Z}_n)$ already vanishes. So pseudo-equivalence extension exists.
- For $k > 1$, $\partial[(p - 1)^{-1}p] \in NK_0(\mathbb{Z}[\mathbb{Z}_n])$ is non-trivial. So pseudo-equivalence extension does not exist. (still obstruction even in ANR category).

3. Semi-Free Action: Example 1

Theorem
If G has element of order p^2 (say G is a p-group and $G \neq \mathbb{Z}_p^{\oplus k}$), then there is no semi-free G-action on homotopy S^1 with $T(p, 1)$ as fixed point.

3. Semi-Free Action: Example 2

If $G = \mathbb{Z}_n$, n is not prime power, then $n = n_1 n_2$, with $n_1, n_2 > 1$ and coprime. Pick $a, b = 1 - a$ satisfying

$$(a, b) = (1, 0) \bmod n_1, \quad (a, b) = (0, 1) \bmod n_2.$$

Then $H = \mathbb{Z}[t, t^{-1}]/(at - b)$ satisfies Smith condition.

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$[at - b] = (0, [a\mathbb{Z}_n], 0, 0)$ in (note $\mathbb{Z}_n = a\mathbb{Z}_n \oplus b\mathbb{Z}_n$)

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No longer obstruction in ANR category!

3. Semi-Free Action: Example 2

Calculate $\partial[a\mathbb{Z}_n]$ by exact sequence

$$\tilde{K}_0(\mathbb{Z}) \oplus \tilde{K}_0(\mathbb{Z}[\xi_n]) \to \tilde{K}_0(\mathbb{Z}_n) \xrightarrow{\partial} K_{-1}(\mathbb{Z}[\mathbb{Z}_n]) \to K_{-1}(\mathbb{Z}) \oplus K_{-1}(\mathbb{Z}[\xi_n]).$$

This is $(K_0(\mathbb{Z}_n) = \oplus_{i=1}^{k} K_0(\mathbb{Z}_{p_i^{m_i}}) = \mathbb{Z}^k$ for $n = p_1^{m_1} \dots p_k^{m_k})$

$$0 \oplus \text{finite} \to \mathbb{Z}^k/\mathbb{Z}(1, \dots, 1) \xrightarrow{\partial} K_{-1}(\mathbb{Z}[\mathbb{Z}_n]) \to 0 \oplus \text{torsionfree}$$

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Take $n_1 = p_1^{m_1}$, $n_2 = p_2^{m_2}\dots p_k^{m_k}$
$\implies [a\mathbb{Z}_n] = [1,0,\dots,0] \neq 0 \in \tilde{K}_0(\mathbb{Z}_n)$
$\implies \partial[a\mathbb{Z}_n] \neq 0 \in K_{-1}(\mathbb{Z}[\mathbb{Z}_n])$.

3. Semi-Free Action: Summary

For $G = \mathbb{Z}_n$ acting on homotopy circle:

- If n is not primer power, then we get K_{-1}-obstruction counterexample.
- If p^2 divides n, then get NK_0-obstruction counterexample.

Thank You