Dirichlet character twisted Eisenstein series and J-spectra

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International Workshop on Algebraic Topology 2019
August 19, 2019

Background

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Definition

Bernoulli numbers B_n are defined to be the coefficients in the Taylor expansion:

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

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$$E_2^{s,t} = H_c^s(\mathbb{Z}_p^\times; \pi_t(K_p^\wedge)) \Longrightarrow \pi_{t-s}\left(S^0_{K(1)}\right).$$

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The generalized Bernoulli numbers $B_{n,\chi}$ associated to χ are defined by:

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with the q-expansion of its normalization given by:

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$$E_k(\gamma \cdot z; \chi) = \chi(a)(cz+d)^k E_k(z;\chi), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

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The automorphic equation above is equivalent to

$$E_{k,\chi} \in \mathrm{Hom}_{(\mathbb{Z}/N)^\times\textit{-rep}}\left(\mathbb{C}_\chi, H^0(\mathcal{M}_{ell}(\Gamma_1(N)), \omega^k)\right).$$

Twisted J-spectra

J-spectra with level structures

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Let $\mathcal{M}_{mult}(N)$ be the moduli stack over $\mathbb{Z}$ of formal groups of height 1 at all primes with μ_N-level structure, that is

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J := S^0_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p S^0_{K/p} \arrow[d] \\
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J(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p S^0_{K/p}\left(p^{v_p(N)}\right) \arrow[d] \\
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Here $S^0_{K/p}(p^v) := \left(K_p^\wedge\right)^{h(1+p^v\mathbb{Z}_p)}$.

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Example

When $N = 7$ and $\chi : (\mathbb{Z}/7)^{\times} \to \mathbb{C}^{\times}$ sending a generator $3 \in (\mathbb{Z}/7)^{\times}$ to $\zeta_6 \in \mathbb{C}^{\times}$. Then $\mathbb{Z}[\chi] \simeq \mathbb{Z}[\zeta_6]$ since $(\mathbb{Z}/7)^{\times} \simeq \mathbb{Z}/6$. This is a free $\mathbb{Z}$-module of rank 2 with basis $\{1, \zeta_6\}$.

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$$\chi(3) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Replacing a homotopy action with a topological one

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One solution

$\mathbb{Z}[\chi] = \mathbb{Z}[\zeta_n]$ for some n. By an obstruction theory of Cooke, the homotopy action of $(\mathbb{Z}/N)^\times$ on $M(\mathbb{Z}[1/n, \zeta_n])$ induced by χ is equivalent to a topological action.

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Some good cases

When $\mathbb{Z}[\chi] = \mathbb{Z}[\zeta_{2^n}]$, the homotopy action on $M(\mathbb{Z}[\chi])$ induced by χ is equivalent to a topological one, e.g. when $N = 2^l \cdot p$ with $p = 2^{2^m} + 1$ for $0 \leq m \leq 4$ being a Fermat prime.

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Construction

Let $\mathbb{Z}[\chi] = \mathbb{Z}[\zeta_n]$, define

$$J(N)^{h\chi} := (J(N) \wedge M(\mathbb{Z}[1/n, \chi^{-1}]))^{h(\mathbb{Z}/N)^\times}.$$

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Remark

$(-)^{h\chi}$ means the homotopy χ-eigen-spectrum.

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$$\left(J(p)^{h\chi}\right)_p^\wedge \simeq \bigvee_{\substack{0 \leq a \leq p-2 \\ \ker \omega^a = \ker \chi}} \left(S^0_{K(1)}(p)\right)^{h\omega^a},$$

where $\omega : (\mathbb{Z}/p)^\times \to \mathbb{Z}_p^\times$ *is the Teichmüller character.*

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Remark

$\left(S^0_{K(1)}(p)\right)^{h\omega^a} \in \mathrm{Pic}^{alg}_{K(1)} \simeq \mathrm{End}(\mathbb{Z}_p^\times)$ corresponds to

$$\mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p)^\times \xrightarrow{\omega^{-a}} \mathbb{Z}_p^\times.$$

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The E_2-page of the HFPSS to compute $\pi_*\left(J(N)^{h\chi}\right)$ can be identified with

$$E_2^{s,t} \simeq \operatorname{Ext}^s_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}(\mathbb{Z}[1/n,\chi],\pi_t(J(N))) \Longrightarrow \pi_{t-s}\left(J(N)^{h\chi}\right),$$

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where $(\mathbb{Z}/N)^\times$ acts on $\mathbb{Z}[1/n,\chi]$ by χ. For p-adic Dirichlet characters, when $N = p$, we further have $\mathbb{Z}_p[\chi] = \mathbb{Z}_p$ and

$$E_2^{s,t} \simeq \mathrm{Ext}^s_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]}\left(\mathbb{Z}_p, \pi_t\left(K_p^\wedge\right)\right) \Longrightarrow \pi_{t-s}\left(\left(S^0_{K(1)}(p)\right)^{h\chi}\right),$$

where $\mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p$ by $\mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p)^\times \xrightarrow{\chi} \mathbb{Z}_p^\times$.

Relations with twisted Eisenstein series

More numeric coincidences

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$$v_p\left(\mathrm{Norm}\left(\frac{2k}{B_{k,\chi^{-1}}}\right)\right) = \begin{cases} v_p(k) + 1, & \text{if } \ker \omega^k = \ker \chi; \\ 0, & \text{else.} \end{cases}$$

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The HFPSS computation shows

$$\pi_{2k-1}\left(\left(J(p)^{h\chi}\right)^{\wedge}_{p}\right) = \begin{cases} \mathbb{Z}/p^{v_p(k)+1}, & \text{if } \ker \omega^k = \ker \chi; \\ 0, & \text{else.} \end{cases}$$

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When χ is trivial, $\ker\omega^k = \ker\chi = (\mathbb{Z}/p)^\times$ iff $(p-1) \mid k$ and we recover the classical numeric coincidence. We will explain this coincidence via congruences of twisted Eisenstein series $E_{k,\chi}$.

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Fix a Dirichlet character $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ with $v_p(N) = v$.

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3. For $M = \mathbb{Z}_p^{\otimes k}[\chi^{-1}]$, use chromatic resolution to show

$$\operatorname*{colim}_m \Big((M/p^m)^{\mathbb{Z}_p^\times}\Big) \simeq \Big(\operatorname*{colim}_m (M/p^m)\Big)^{\mathbb{Z}_p^\times} \simeq H_c^1(\mathbb{Z}_p^\times; M).$$

Congruence and group cohomology

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Proposition

Let M *be a* $\mathbb{Z}_p^\times$*-representation in finite free* $\mathbb{Z}_p$*-modules with no non-zero fixed points, then* $H_c^1(\mathbb{Z}_p^\times; M) \simeq \operatorname{colim}_m \left((M/p^m)^{\mathbb{Z}_p^\times} \right)$.

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Apply $H_c^*(\mathbb{Z}_p^\times; -)$ to the short exact sequence:

$$0 \longrightarrow M \longrightarrow p^{-1}M \longrightarrow M/p^\infty \longrightarrow 0,$$

we get an isomorphism $(M/p^\infty)^{\mathbb{Z}_p^\times} \simeq H_c^1(\mathbb{Z}_p^\times; M)$.

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we get an isomorphism $(M/p^\infty)^{\mathbb{Z}_p^\times} \simeq H_c^1(\mathbb{Z}_p^\times; M)$. The claim now follows from the isomorphism

$$\operatorname*{colim}_m \left((M/p^m)^{\mathbb{Z}_p^\times} \right) \xrightarrow{\sim} \left(\operatorname*{colim}_m M/p^m \right)^{\mathbb{Z}_p^\times} \simeq (M/p^\infty)^{\mathbb{Z}_p^\times}.$$

$\square$

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Let κ be a perfect field of characteristic p. Let A be a flat $\mathbb{W}(\kappa)$-algebra such that A/p is an integrally closed domain over κ and that A admits an endomorphism $\varphi : A \to A$ that lifts the Frobenius φ_0 on A/p (the p-th power map).

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Theorem

The following categories are equivalent:

$$\begin{tikzcd}
\left\{\begin{array}{c}\textit{Projective } A\textit{-modules } M \textit{ of}\\ \textit{rank } r \textit{ with } F : \varphi^* M \xrightarrow{\sim} M\end{array}\right\} \arrow[r, dash, "\mathrm{I}"] \arrow[d, dash, "\mathrm{V}"'] & \left\{\begin{array}{c}\textit{Continuous } \pi_1^{\acute{e}t}(A)\\ \textit{actions on } \mathbb{Z}_p^{\oplus r}\end{array}\right\} \arrow[d, dash, "\mathrm{III}"] \arrow[dl, dash, "\mathrm{II}"] \\
\left\{\begin{array}{c}p\textit{-divisible formal groups over } \mathrm{Spf}\, A\\ \textit{of dimension and height } r\end{array}\right\} \arrow[r, dash, "\mathrm{IV}"'] & \left\{\begin{array}{c}\mathrm{GL}_r(\mathbb{Z}_p)\textit{-torsors}\\ \textit{over } \mathrm{Spf}\, A\end{array}\right\}
\end{tikzcd}$$

Congruences in the RH correspondence

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Let $\widehat{G}$ be a one-dimensional formal group of height 1 over A. Denote the Dieudonné module associated to $\widehat{G}$ by $(M, F : \varphi^* M \xrightarrow{\sim} M)$ and the Galois descent data by $\rho \in H^1(\pi_1^{ét}(A); \mathbb{Z}_p^\times)$.

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1. $\widehat{G}[p^m] \simeq \mu_{p^m}$.
2. *There is a generator* $\gamma \in M$ *such that* $F\gamma \equiv \gamma \mod p^m$.
3. ρ *is trivial mod* p^m, *i.e. the image of* $\rho : \pi_1^{ét}(A) \to \mathbb{Z}_p^\times$ *is contained in* $1 + p^m\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$.

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In particular, when $m = \infty$, *the followings are equivalent:*

1. $\widehat{G} \simeq \widehat{G}_m$.
2. *There is a generator* $\gamma \in M$ *such that* $F\gamma = \gamma$.
3. ρ *is the trivial representation.*

Dirichlet equivariance and Galois descent

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Construction

As the invertible sheaf $\omega^{\otimes k}$ over $\mathcal{M}_{ell}^{ord}(p^v)$ is the pullback of the invertible sheaf $\omega^{\otimes k}$ over $\mathcal{M}_{ell}^{ord}$, there is a canonical isomorphism

$$f_\sigma : \omega^{\otimes k} \xrightarrow{\sim} \sigma^* \omega^{\otimes k}, \quad \sigma \in \mathrm{Aut}_{\mathcal{M}_{ell}^{ord}}(\mathcal{M}_{ell}^{ord}(p^v)) \simeq (\mathbb{Z}/p^v)^\times ,$$

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$$1 \otimes \chi^{-1}(\sigma) : \omega^{\otimes k} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi^{-1}] \xrightarrow{\sim} (\sigma \otimes 1)^* (\omega^{\otimes k} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi^{-1}]).$$

Denote the resulting sheaf over $\mathcal{M}_{ell}^{ord}$ by $\mathscr{F}_{k,\chi}$.

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Lemma

$$\mathrm{Hom}_{\mathbb{Z}_p[(\mathbb{Z}/p^v)^\times]}\left(\mathbb{Z}_p[\chi], H^0(\mathcal{M}_{ell}^{ord}(p^v), \boldsymbol{\omega}^{\otimes k})\right) \simeq H^0(\mathcal{M}_{ell}^{ord}, \mathscr{F}_{k,\chi}).$$

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$$\rho_{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}) \xrightarrow{\rho} \mathbb{Z}_p^\times \xrightarrow{(-)^k \cdot \widetilde{\chi}^{-1}} (\mathbb{Z}_p[\chi])^\times \hookrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(\mathbb{Z}_p[\chi]),$$

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Proposition

$E_{k,\chi} \equiv 1 \mod I \trianglelefteq \mathbb{Z}_p[\chi]$ *iff the* $\mathbb{Z}_p^\times$*-representation* $\mathbb{Z}_p^{\otimes k} \otimes \mathbb{Z}_p[\chi^{-1}]$ *is trivial mod* I.

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2. *Over $\mathcal{M}_{ell}^{ord}(p^v)$, there is $\gamma \in \boldsymbol{\omega}^{\otimes k} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi]$ such that:*
   - *The γ generates the module $\boldsymbol{\omega}^{\otimes k} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi]/I$ over $\mathcal{M}_{ell}^{ord}(p^v) \times_{\operatorname{Spf} \mathbb{Z}_p} \operatorname{Spf} \mathbb{Z}_p[\chi]$;*
   - *For any $g \in (\mathbb{Z}/p^v)^\times$, $g \cdot \gamma = \chi(g)\gamma$;*
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4. *The $\mathbb{Z}_p^\times$-representation $\mathbb{Z}_p^{\otimes k}[\chi^{-1}]$ is trivial mod I.*
5. *There is a surjection $H^1\left(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes k}[\chi^{-1}]\right) \twoheadrightarrow \mathbb{Z}_p[\chi]/I$.*

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Moreover, $E_{k,\chi} \equiv 1 \mod I \trianglelefteq \mathbb{Z}_p[\chi]$ *is the maximal congruence iff* $H^1\left(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes k}[\chi^{-1}]\right) \simeq \mathbb{Z}_p[\chi]/I$.

Thanks for your attention!