Equivariant Factorization Homology and Nonablian Poincaré Duality

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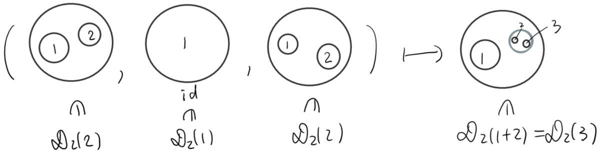
History

- Belinson-Drinfeld;
- Lurie, Ayala-Francis;
- Kupers-Miller, Knudsen, ...

little n-disk operad

The operad $\mathscr{D}_n$ has the following data:

- Spaces $\mathscr{D}_n(k) = \{e_1, \cdots, e_k |$ conditions $\}$.
  - Each $e_i : D^n \to D^n$ is in the form of $e_i(\mathbf{v}) = a\mathbf{v} + \mathbf{b}$ for $a > 0, \mathbf{b} \in D^n$;
  - The images of e_i's are disjoint;
- Struture maps $\gamma : \mathscr{D}_n(k) \times \mathscr{D}_n(j_1) \times \cdots \times \mathscr{D}_n(j_k) \to \mathscr{D}_n(j_1 + \cdots + j_k)$.



G-operad

- G: finite group. V: n-dimensional orthogonal G-representation.
- A G-operad in Top is an operad such that the spaces are G-spaces and structure maps are G-equivariant. Equivalently, it is an operad in Top^G.

Notation

- $G\mathrm{Top}$ is the category of G-spaces and *non-equivariant* maps;
- Top^G is the category of G-spaces and equivariant maps;
- $G\mathrm{Top}$ is enriched in Top^G.

Example

1. Let X be an object in a Top^G-enriched category $(\mathcal{C}, \otimes)$, then $\mathrm{End}_X^{\otimes}(k) = \mathrm{Hom}_{\mathcal{C}}(X^{\otimes k}, X)$ is a G-operad.
2. Replacing the disk D^n by the unit disk in V, we get the little V-disk operad $\mathscr{D}_V$ (Guillou-May).

E_n-algebra

- The (reduced) operad $\mathscr{D}_n$ is associated with a monad $\mathrm{D}_n : \mathrm{Top}_* \to \mathrm{Top}_*$:

$$\mathrm{D}_n X = \coprod_k \mathscr{D}_n(k) \times_{\Sigma_k} X^k / \sim$$

- An algebra over $\mathscr{D}_n$ is space A with structure maps

$$\lambda : \mathscr{D}_n(k) \times_{\Sigma_k} A^k \to A$$

  that satisfies the unital, associativity and Σ-equivariant diagrams.
- Equivalently, it is an algebra over D_n with structure maps

$$\lambda : \mathrm{D}_n A \to A.$$

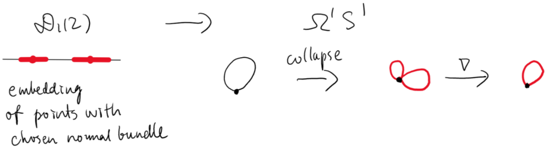
- Such an algebra is called an E_n-algebra.

E_n-algebra

Example

$\Omega^n X$ is an E_n-algebra.

$$\mathrm{D}_n(\Omega^n X) \xrightarrow{s(\Omega^n X)} \Omega^n \Sigma^n (\Omega^n X) \xrightarrow{\text{counit}} \Omega^n X.$$



E_n-algebra

One alternative way to see an E_n-algebra A:

- Let $\mathrm{Disk}_n^{\mathrm{fr}}$ be the symmetric monoidal topological category with

$$\begin{aligned} \text{obj} &: [k] \text{ for } k \geq 0; \\ \text{mor} &: \mathrm{Emb}^{\mathrm{fr}}(\sqcup_k D^n, \sqcup_l D^n); \\ \otimes &: [k] \otimes [l] \cong [k+l]. \end{aligned}$$

- Then A is a symmetric monoidal topological functor $\mathrm{Disk}_n^{\mathrm{fr}} \to \mathrm{Top}$.

factorization homology for framed manifold

Factorization homology of framed manifolds with coefficient A is the symmetric monoidal topological left Kan extension:

$$
\begin{array}{ccc}
\mathrm{Disk}_n^{\mathrm{fr}} & \xrightarrow{A} & \mathcal{C} \\
\downarrow & \nearrow_{\int_{\_} A} & \\
\mathrm{Mfld}_n^{\mathrm{fr}} & &
\end{array}
$$

the Top^G-category $\mathrm{Mfld}_n^{\mathrm{fr}_V}$

Definition

A smooth G-manifold is V-framed if there is G-vector bundle isomorphism

$$\mathrm{T}M \cong M \times V.$$

Example

1. V is V-framed;
2. $G = C_2$. Let σ be the sign representation. Then S^σ is σ-framed.
3. $G = C_p$. Then S^1_{rot} is $\mathbb{R}$-framed.
4. $G = C_p$. Let λ be the 2-dimensional rotation representation. Then $S^1_{\mathrm{rot}} \times \mathbb{R}$ is both λ- and $\mathbb{R}^2$-framed.

Construction

Following Kupers-Miller, we construct a symmetric monoidal Top^G-category $(\mathrm{Mfld}_n^{\mathrm{fr}_V}, \sqcup)$ of V-framed manifolds and V-framed embeddings such that $\mathrm{Emb}^{\mathrm{fr}_V}(V, V) \simeq *$.
(Idea: Use Steiner paths.)

- Endomorphism operad $\mathscr{D}_V^{\mathrm{fr}_V}$ and monad $\mathrm{D}_V^{\mathrm{fr}_V}$. ($\mathscr{D}_V^{\mathrm{fr}_V}$ is equivalent to $\mathscr{D}_V$.)
- Moreover, any manifold M gives rise to a functor

$$\begin{aligned} \mathrm{D}_M^{\mathrm{fr}_V} : \quad \mathrm{Top}^G &\to \mathrm{Top}^G \\ X &\mapsto \textstyle\coprod_{k\geq 0} \mathrm{Emb}^{\mathrm{fr}_V}(\sqcup_k V, M) \times_{\Sigma_k} X^k / \sim . \end{aligned}$$

- $\mathrm{D}_M^{\mathrm{fr}_V} X$ is the V-fattened configuration space on M with based labels in X.

Proposition

Evaluation at 0 gives a G-homotopy equivalence

$$ev_0 : \mathrm{D}_M^{\mathrm{fr}_V} X \to \coprod_k \mathrm{PConf}(M, k) \times_{\Sigma_k} X^k / \sim .$$

structures

$$\mathrm{D}_M^{\mathrm{fr}_V}(X) = \coprod_{k\geq 0} \mathrm{Emb}^{\mathrm{fr}_V}(\sqcup_k V, M) \times_{\Sigma_k} X^k / \sim .$$

- Composition $\mathrm{D}_M^{\mathrm{fr}_V} \circ \mathrm{D}_V^{\mathrm{fr}_V} \to \mathrm{D}_M^{\mathrm{fr}_V}$; $\mathrm{D}_V^{\mathrm{fr}_V} \circ \mathrm{D}_V^{\mathrm{fr}_V} \to \mathrm{D}_V^{\mathrm{fr}_V}$;
- Unit $\mathrm{Id} \to \mathrm{D}_V^{\mathrm{fr}_V}$ from the element $\mathrm{id} : V \to V$;

Take a (non-degenerately based) $\mathrm{D}_V^{\mathrm{fr}_V}$-algebra A in Top^G,

- Struture map $\mathrm{D}_V^{\mathrm{fr}_V}(A) \to A$.

We have a simplicial G-space:

$$\mathbf{B}_\bullet(\mathrm{D}_M^{\mathrm{fr}_V}, \mathrm{D}_V^{\mathrm{fr}_V}, A) = \mathrm{D}_M^{\mathrm{fr}_V}(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet(A).$$

Definition

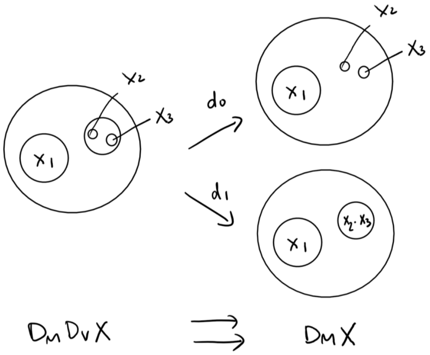
The factorization homology of M with coefficient A is

$$\int_M A := \mathbf{B}(\mathrm{D}_M^{\mathrm{fr}_V}, \mathrm{D}_V^{\mathrm{fr}_V}, A).$$

structure

$$\int_M A := \mathbf{B}(\mathrm{D}_M^{\mathrm{fr}_V}, \mathrm{D}_V^{\mathrm{fr}_V}, A).$$

The bar construction is a model for configuration spaces with E_V-summable labels (Salvatore).



scanning map

- The scanning maps on configuration spaces have been studied by McDuff, Segal, Bödigheimer, Manthorpe-Tillmann, ...
- It maps a configuration of points on M to a section of $\mathrm{T}M$. Intuitivly, it is the Pontryagin-Thom collapse map.

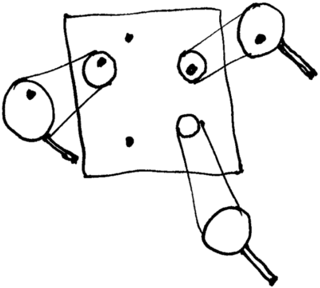


Figure: illustration of the scanning map by Church

- It maps a configuration of points on M to a section of $\mathrm{T}M$.
- In our V-framed case, it takes the form:

Construction

$$s : \mathrm{D}_M^{\mathrm{fr}_V}(X) \to \mathrm{Map}_*(M^+, \Sigma^V X).$$

- For labelled configuration space on a G-manifold M, the following theorem has been proved geometricly: (for $M = V$, it is the equivariant recognition priciple by Guillou-May)

Theorem (Rourke-Sanderson)

The scanning map is a G-weak equivalence if X is G-connected.

$$s : \mathrm{D}_M^{\mathrm{fr}_V}(X) \to \mathrm{Map}_*(M^+, \Sigma^V X).$$

The scanning map is simplicial:

$$s : \mathrm{D}_M^{\mathrm{fr}_V}(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet(X) \to \mathrm{Map}_*(M^+, \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet X).$$

So it realizes to

$$\int_M A = \mathrm{D}_M^{\mathrm{fr}_V}(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet(A) \to |\mathrm{Map}_*(M^+, \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A)|$$
$$\to \mathrm{Map}_*(M^+, |\Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A|) = \mathrm{Map}_*(M^+, \mathbf{B}^V A).$$

Nonabelian Poincaré duality

Theorem (Z.)

Let M be a V-framed manifold and A be a $\mathrm{D}_V^{\mathrm{fr}_V}$-algebra in Top^G. Assume that A is non-degenerately based and G-connected. Then the scanning map induces a G-weak equivalence:

$$\int_M A \to \mathrm{Map}_*(M^+, \mathbf{B}^V A).$$

Application: baby equivariant Poincaré duality

Let A be a discrete $\mathbb{Z}[G]$-module. Then it is a G-E_∞-space.

$$\int_M A = M \otimes A.$$

The equivariant Dold-Thom theorem:

Theorem (Lima-Filho, Santos)

$$\pi_\bigstar^G(X \otimes A) \cong \tilde{\mathrm{H}}_\bigstar^G(X, \underline{A}).$$

Corollary

For V-framed manifold M, there is isomorphism:

$$\tilde{\mathrm{H}}_\bigstar^G(M, \underline{A}) \cong \mathrm{H}_G^{V-\bigstar}(M^+, \underline{A}).$$

Application: factorization homology on Thom spectra

Theorem (Horev-Klang-Z.)

Let A be the Thom spectrum of an E_{V+1}-map $\Omega^{V+1}X \to \mathrm{Pic}(\mathrm{Sp}^G)$ such that X is suitably connected. Then

$$\int_{S^V \times \mathbb{R}} A \simeq A \wedge \Omega X_+.$$

Take $G = C_2$, σ: the sign representation, $\rho \cong \sigma + 1$: the regular representation.

Theorem (Behrens-Wilson)

The Eilenberg-MacLane spectrum $\mathrm{H}\underline{\mathbb{F}_2}$ is equivariantly the Thom spectrum of a ρ-fold loop map $\Omega^\rho S^{\rho+1} \to B_{C_2}O$.

Corollary

$$\mathrm{THR}(\mathrm{H}\underline{\mathbb{F}_2}) \simeq \int_{S^\sigma} \mathrm{H}\underline{\mathbb{F}_2} \simeq \mathrm{H}\underline{\mathbb{F}_2} \wedge (\Omega S^{\rho+1})_+.$$

Questions:

- As a functor for M: can we get a better understood equivariant Poincaré duality theorem?
- As a functor for A: can we get useful invariants for algebras with partial norms?
- For a ring spectrum R, can we identify R-orientable manifold and $E_n^{R-\mathrm{ori}}$-algebra?
- We need to study general tangential structures.

Tangential structure

- $B_G O(n)$: the classifying space for G-equivariant n-dimensional vector bundle.
- Tangential structure: a map $\theta : B \to B_G O(n)$.
- θ-framing on M: a G-bundle map $\phi : \mathrm{T}M \to \theta^*\gamma$, where γ is the universal bundle on $B_G O(n)$.
- Equivalently,

$$
\begin{array}{ccc}
 & & B \\
 & \overset{\tau_B}{\nearrow} \;\Uparrow & \downarrow \theta \\
M & \xrightarrow{\tau} & B_G O(n)
\end{array}
$$

E_V^θ-algebra

Let θ be a tangential structure such that V is θ-framed.
We can identify $\mathscr{D}_V^\theta$ with a semidirect product of $\mathscr{D}_V$ (Salvatore-Wahl):

Proposition

There is an equivalence of G-operads: $\mathscr{D}_V^\theta \simeq \mathscr{D}_V \rtimes (\mathrm{Emb}^\theta(V, V))$.
(Here, $\mathrm{Emb}^\theta(V, V)$ is a group object in Top^G. It is equivalent to ΩB.)

In terms of algebras:

$$(\mathrm{Top}^G)^\Pi \cong \mathrm{Top}^{\Pi \rtimes_\alpha G}.$$

$$\mathscr{C}[\mathrm{Top}^G] \cong (\mathscr{C} \rtimes G)[\mathrm{Top}].$$

$$(\mathscr{C} \rtimes \Pi)[\mathrm{Top}^G] \cong \mathscr{C}[\mathrm{Top}^{\Pi \rtimes_\alpha G}] \cong \mathscr{C} \rtimes (\Pi \rtimes_\alpha G)[\mathrm{Top}].$$

Thank you!