Talk 5, 6
Basics of Spectra

Hana Jia Kong

The University of Chicago

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“The first category of spectra”

- (Fraudenthal’s suspension theorem)
  Let Y be $(n-1)$-connected space. Then $[X,Y] \to [\Sigma X, \Sigma Y]$ is iso if $\dim(X) < 2n-1$, and surjection if $\dim(X) = 2n-1$.
- SW category (Spanier–Whitehead, 1953):
  - objects: pointed CW-cplx
  - morphisms: $\mathrm{Hom}(X,Y) := \varinjlim_q [\Sigma^q X, \Sigma^q Y]$
- More generally, we can include the non-connective objects:
  - objects: (X, n) $X \in$ pointed CW-cplx, $n \in \mathbb{Z}$
  - morphisms: $\mathrm{Hom}((X,n),(Y,m)) := \varinjlim_q [\Sigma^{q+n} X, \Sigma^{q+m} Y]$
- Pros: ① Convenient for duality
  ② Triangulated category
- Cons: ① Don't have all coproducts
  ② $\mathrm{Ho}(\mathrm{Top}_*) \to SW$ doesn't preserve coproducts.

Recall: how did spectra arise

- Roughly speaking, a spectrum consists of the following data:
  - $E_n \in Top_*$, $n \in \mathbb{Z}$
  - structure maps: $\sigma_n : \Sigma E_n \to E_{n+1}$
- They represent cohomology theories.

Theorem (Brown, 1962)

$\hat{E}^*(-)$ reduced generalized cohomology theory.

Then there exists an Ω-spec s.t.

$$\hat{E}^n(X) \cong [X, E_n]$$

adjoint of σ_n $E_n \to \Omega E_{n+1}$ is a w.e.

- Examples: HA, KU, MU, S^0

Why spectra

- Spectra and cohomology theories they are not equivalent. There exist "hyperphantom maps": $f: E \to F$ map of spectra s.t.
  ① $f \not\simeq 0$
  ② f induces 0 on cohomology theories.
- The category has point-set models to record the geometric data.
- Has good structures after passing to the homotopy theory.

Multiplicative enhancement

- Multiplicative cohomology theories: In many examples, we have

  $\hat{E}^n(X) \otimes \hat{E}^m(Y) \longrightarrow \hat{E}^{m+n}(X \wedge Y)$

  - cup product
  - tensor of vector bundles (KU)
  - cartesian product of manifolds (MU)

- In view of Brown's theorem:

  $[X, E_n] \otimes [Y, E_m] \longrightarrow [X \wedge X, E_n \wedge E_m] \xrightarrow{\text{diag}} [X, E_n \wedge E_m]$

  $\xrightarrow{\mu_{n+m}} [X, E_{n+m}]$

Towards a good smash product

- Want a good symmetric monoidal category of spectra.
  - (Aside) Symmetric monoidal category $(\mathscr{C}, \otimes, U)$:
    - Tensor: $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$
    - Unit: $U \in \mathscr{C}$
    - unital & associative & commutative diagrams.
    - Closed symmetric monoidal: $\mathrm{Hom}_{\mathscr{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathscr{C}}(A, \underline{\mathrm{Hom}}(B, C))$
- A naive try: $(X \wedge Y)_n \overset{?}{:=} X_n \wedge Y_n$ problem: str. map hard to define $\Sigma (X \wedge Y)_n \to (X \wedge Y)_{n+1}$
- Adams' "handcrafted" $\wedge$:
  commutative & associative up to homotopy
- $\mathrm{Ho}(S)$: closed sym monoidal category.

An obstacle: Lewis' theorem

Theorem (Lewis, 1990)

There is no symmetric monoidal category $(S, \wedge, U)$ of spectra with the following properties:

- There exists a lax monoidal adjunction $\Sigma^\infty : Top_* \leftrightarrows S : \Omega^\infty$.
- The canonical map $\Sigma^\infty S^0 \to U$ is an isomorphism.
- Let Q be the stabilization functor $QX := \mathrm{colim}_n \Omega^n \Sigma^n(X)$. There is a natural weak homotopy equivalence f:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\iota"'] & \Omega^\infty \Sigma^\infty(X) \arrow[d, "f"] \\ & QX \end{tikzcd}.$$

A sketch proof of Lewis' theorem

1) Adjunction : $\Sigma^\infty \dashv \Omega^\infty$
2) $\Sigma^\infty S^0$ is the unit
3) $\Omega^\infty \Sigma^\infty X \xrightarrow{\simeq} QX$

Sketch proof:

- $\Sigma^\infty S^0$ is the unit $\Rightarrow$ $\Sigma^\infty S^0$ is a comm. monoid.
- Ω^∞ lax sym monoidal $\Rightarrow$ $\Omega^\infty \Sigma^\infty S^0$ is a comm. monoid.
- $\Omega^\infty \Sigma^\infty S^0 \simeq QS^0 \Rightarrow QS^0$ is comm. monoid

$\xRightarrow{\text{Moore's th'm}}$ $QS^0 \simeq \prod_\alpha HA_\alpha$ contradiction!

Compromises: Different models

- S-modules $\mathscr{M}_S$ (Elmendorf–Kriz–Mandell–May, 1997)
- Orthogonal spectra $Sp^{\mathbf{O}}$(Mandell–May–Schwede–Shipley, 1998)
- Symmetric spectra Sp^{Σ}(Hovey–Shipley–Smith,2000)

Orthogonal spectra $Sp^{\mathbf{O}}$

space of all linear isometric embedding of $V \to W$

- Topological category $\mathbf{O}$:
  - objects: finite dim inner product vector spaces
  - morphisms: $\mathcal{O}(V,W) := Th(\xi(V,W))$ vector bundle over $L(V,W)$
    $\{(w,\varphi) \mid w \in W,\ w \perp \varphi(V)\}$
- Definition: An orthogonal spectra is a continuous functor $E : \mathcal{O} \to Top_*$. Morphisms are natural transformations.
- Smash product:

  Day convolution.

$$\begin{tikzcd} \mathcal{O}\times\mathcal{O} \arrow[r, "E\bar{\wedge}F"] \arrow[d, "\oplus"'] & Top_* \\ \mathcal{O} \arrow[ur, dashed, "\text{left Kan extension}"'] & \end{tikzcd}$$

$\rightsquigarrow E\wedge F$ (the dotted arrow) (has concrete formula)

- Weak equivalences:
  Stable π_*-equivalences : $\pi_n(E) := \operatorname{colim}_V [S^{\dim(V)+n}, E(V)]$
- The third property in Lewis' theorem fails.
  $\Sigma^\infty S^0 = \{S^0, S^1, S^2, S^3, \dots\}$

EKMM spectra

- Idea: make things coordinate-free. Fix a universe $U \cong \mathbb{R}^\infty$.
- Definition: A prespectrum E is the structure consisting of
  - $E(V) \in \mathrm{Top}_*$, $\forall$ finite dim inner product v. s. $V \subseteq U$
  - $\sigma_{V,W} : \Sigma^{W-V} E(V) \longrightarrow E(W)$, $\forall V \subseteq W$
- Definition:

  E is a (Lewis–May–Steinberger) spectrum if $\sigma_{V,W}$ is a homeomorphism, $\forall\ V \subseteq W \subseteq U$.
- Adjunctions:

$$\begin{tikzcd} \mathrm{Top}_* \arrow[r, shift left, "\Sigma^\infty"] \arrow[r, phantom, "\perp"] & PS \arrow[l, shift left, "\Omega^\infty"] \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & SU \arrow[l, shift left, "\ell"] \end{tikzcd}$$

spectrification

forgetful functor

EKMM smash product: the first try

$$
\begin{array}{ccc}
W & \longleftarrow\!\shortmid & (V, V') \\
\mathcal{U} & \xleftarrow[\cong]{\alpha} & \mathcal{U} \oplus \mathcal{U}
\end{array}
$$

- The first try: $E \wedge_\alpha F(W) = \underbrace{E \bar{\wedge} F}_{\hookrightarrow \text{ indexed over } \mathcal{U} \oplus \mathcal{U}}(V, V') = E(V) \wedge F(V')$
- Problem: Depends on α
- Fix?: The space of all $\mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$ is contractible.

  $\rightsquigarrow$ use all choices
- Linear isometry operad $\mathscr{L}(n) := \mathcal{L}(\mathcal{U}^{\oplus n}, \mathcal{U})$

  $\mathcal{L}(2)$ = the space of all choices $\mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$

EKMM smash product: the second try

- Idea: replace a choice $\{U \oplus U \to U\} \in \mathscr{L}(2)$ by all choices $\mathscr{L}(2) = \mathscr{L}(U \oplus U, U)$.
- The second try: (twisted half smash product)

  $$\mathcal{L}(2) \ltimes (E \bar{\wedge} F) := \bigcup_{\alpha \in \mathcal{L}(2)} E \wedge_\alpha F$$

- Problem: non associative.

  $$\mathcal{L}(1) \ltimes E := \bigcup_{\alpha \in \mathcal{L}(1)} E$$

- Fix?: Quotient out the duplicated copies

  $$E \wedge_{\mathcal{L}} F = \mathcal{L}(2) \ltimes_{\mathcal{L}(1) \times \mathcal{L}(1)} (E \bar{\wedge} F)$$

-
  - $\underset{U \oplus U \to U}{\overset{f}{\mathcal{L}(2)}} \times \underset{U \to U}{\overset{g}{\mathcal{L}(1)}} \times \underset{U \to U}{\overset{h}{\mathcal{L}(1)}} \to \mathcal{L}(2).$ $\quad \mapsto f \circ (g \oplus h)$
  - need $\mathcal{L}(1) \curvearrowright E$, $\mathcal{L}(1) \curvearrowright F$
- Definition:

  An $\mathbb{L}$-spec is LMS spec with an action of $\mathcal{L}(1)$.

EKMM smash product: the last step

$\mathscr{M}_S$ S-modules

$\shortparallel$

- Idea: make $\wedge_{\mathscr{L}}$ unital by restricting to $\{$all E with $E \wedge_{\mathscr{L}} S \cong E$. $\}$
- Comparison: $\subset$

$$\begin{tikzcd}
Top_* \arrow[r, shift right, "\Sigma^\infty"'] \arrow[rrrr, bend right, "\Sigma^\infty"'] & P\mathscr{S} \arrow[l, shift right, "\Omega^\infty"'] \arrow[l, phantom, "\perp"] \arrow[r, shift right, "\mathbb{L}"'] & \mathscr{S} \arrow[l, shift right, "\ell"'] \arrow[l, phantom, "\perp"] \arrow[r, shift right, "\mathbb{L}(1) \ltimes (-)"'] & \mathbb{L}[\mathscr{S}] \arrow[l, shift right, "\ell"'] \arrow[l, phantom, "\perp"] \arrow[r, shift right, "S \wedge_{\mathscr{L}} (-)"'] & \mathscr{M}_S \arrow[l, shift right, "F_{\mathscr{L}}(S,-)"'] \arrow[l, phantom, "\perp"]
\end{tikzcd}$$

$\mathbb{L}$-Spectra

T

- Weak equivalences: Stable π_*-eq
- The third property in Lewis' theorem fails.

(ref: compromises forced by Lewis' th'm Elmendorf).

Symmetric spectra Sp^Σ

- Definition: The category of symmetric spectra
  - objects: Equivariant spaces $X_n \circlearrowleft \Sigma_n$, $\forall n \in \mathbb{N}$, str. maps $\Sigma X_n \to X_{n+1}$ with equivariance
  - morphisms: $\underline{\text{equivariant}}$ maps levelwise.

$\Sigma^m X_n \to X_{m+n}$ is $\Sigma_m \times \Sigma_n$ equivariant.

- Smash product:

$$\begin{tikzcd} \Sigma_p \times \Sigma_q \arrow[r, "X \bar{\wedge} Y"] \arrow[d] & Top_* \\ \Sigma_{p+q} \arrow[r, dotted] & X \otimes Y \arrow[u, dotted] \end{tikzcd}$$

  - The first step: $X \otimes Y$
    By Day convolution
  - Key fact: ① $\Sigma^\infty S^0 =: S$ is a comm monoid in $(Sp^\Sigma, \otimes, S)$
    ② $\{\text{Sym Spec}\} \simeq \{\text{modules over } S\}$
  - $X \wedge Y := X \otimes_S Y$
- Warning: If we take stable π_* to be the w.e. $\Rightarrow$ too many ~~homotopy~~ type.

  Need to take weaker replacement.
- The third property in Lewis' theorem fails.

Comparison

Differences:

- Orthogonal spectra $Sp^{\mathbf{O}}$: model (modified) for Hill–Hopkins–Ravenel
- EKMM spectra $\mathscr{M}_S$: hard to define

  Ω^∞ records 0th space information

  All obj are fibrant   S is not cofibrant
- Symmetric spectra Sp^Σ: Stable π_* not w.e.

  (hard to do equivariant thry)

  (Shipley) ∃ convenient model str. on comm. ring objs.

Quillen equivalences:

- (MMSS) $Sp^\Sigma \xleftrightarrow[\text{of Q.e.}]{\text{zigzag}} Sp^O$
- (Schwede) $Sp^\Sigma \xleftrightarrow[\text{Q.e.}]{} \mathcal{M}_S$
- (Schwede–Shipley) $Sp^\Sigma \xleftrightarrow[\text{Q.e.}]{}$ any 'correct' cat of spec

Some properties

- $\mathrm{Ho}(S)$ is triangulated.
  - (Aside) Triangulated category $(\mathscr{C}, [1])$:
    - translation: $[1] : \mathscr{C} \to \mathscr{C}$ + axioms
    - distinguished triangles: $\{X \to Y \to Z \to X[1]\}$
    - An familiar example: $D(\mathbb{Z})$
- Fiber sequences are cofiber sequences and vice versa.

The ∞-treatment: the abstract definition

- $\mathscr{S}_*$: the ∞-category of spaces.
- Definition: $\Omega X :=$ $\lim \left(\begin{array}{ccc} & & * \\ & & \downarrow \\ * & \to & X \end{array} \right)$ $\qquad \Omega : \mathcal{S}_* \to \mathcal{S}_*$
- Definition: The ∞-category Sp is

  $$\varprojlim \left(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right)$$

- Works more generally for $\mathscr{C}$ with finite limits

  1) $\mathcal{C}_* := \mathcal{C}^{*/}$ the pointed cat

  2) define $\Omega : \mathcal{C}_* \to \mathcal{C}_*$

  2) take $\mathrm{Sp}(\mathcal{C}) := \varprojlim \left(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$

A concrete construction via excisive functors

We have a more concrete construction via excisice functors

- $\mathscr{C}$: an ∞-category with finite limits.
- $\mathscr{S}_*^{\mathrm{fin}}$: ∞-cat of finite spaces
- Definition:

  A spectrum object in $\mathscr{C}$ is a functor $F : \mathscr{S}_*^{\mathrm{fin}} \to \mathscr{C}$ such that is
  - excisive: sends pushout squares to pullback squares
  - reduced: sends terminal objs to terminal objs

(Continued)

- When $F : \mathscr{S}_*^{\mathrm{fin}} \to \mathscr{C}$ reduced and excisive:

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \quad \overset{F}{\rightsquigarrow} \quad \begin{tikzcd} F(S^n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\ * \arrow[r] & F(S^{n+1}) \end{tikzcd} \rightsquigarrow F(S^n) = \Omega F(S^{n+1})$$

Compare $E_n \xrightarrow{\simeq} \Omega E_{n+1}$

- Properties(Lurie, Higher Algebra):
  - Adjunction: $\mathcal{C}$ presentable. Ω^∞ admits a left adjoint:
    $$\Omega^\infty : Sp(\mathcal{C}) \rightleftarrows \mathcal{C} : \Sigma^\infty.$$
  - Universal property: $\mathcal{C}, \mathcal{D}$ presentable, $\mathcal{D}$ stable.
    $$\mathrm{Pr}^L(Sp(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Pr}^L(\mathcal{C}, \mathcal{D})$$ + the dual statement.
  - Agrees with the model category definition.

Stable ∞-category

- Definition: An ∞-category $\mathscr{C}$ is stable if:
    - has zero obj
    - Every morph admits a fiber & cofiber
    - fib seq $\Leftrightarrow$ a cof seq
- (Lurie) $\mathscr{C}$: a stable ∞-category. $\mathrm{Ho}(\mathscr{C})$ has the structure of a triangulated category.

Thank you!