Equivariant Spaces

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July 20, 2021

1 G-spaces and G-CW complexes

The main objects in equivariant homology theory and homotopy theory are G-spaces which are spaces equipped with an action by a topological group G. As mentioned for nonequivarint spaces, we take all spaces to be compactly generated and weak Hausdorff.

Definition 1.1. A (left) G-space is a topological space X with continuous actions $G \times X \to X$ such that $ex = x$ and $g(g'x) = (gg')x$.

Remark 1.2. Right G-spaces can be considered as left G-spaces by setting $gx = x(g^{-1})$, and vice versa.

Definition 1.3. A G-map $f : X \to Y$ is a continuous map f such that $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. We use $\mathrm{Map}_G(X, Y)$ to denote the space of all G-maps. We also call them equivariant maps.

They togeother form the category of G-spaces, $G\mathrm{Top}$. We also have the category Top_G of G-spaces and all (non-equivariant) continuous maps.

The usual constructions on spaces apply equally well in this category. In particular, we have Cartesian product $X \times Y$ with G acting diagonally: which means $g(x, y) = (gx, gy)$. The space of all continous maps from X to Y, $\mathrm{Map}(X, Y) = Y^X$ is a G-space too. The action is $(g \cdot f)(x) = gf(g^{-1}x)$. Note that $\mathrm{Map}_G(X, Y) = \mathrm{Map}(X, Y)^G$. We will introduce fixed points in the next section.

$(G\mathrm{Top}, \times, \mathrm{pt})$ is a closed Cartesian monoidal category with the internal hom being $\underline{G\mathrm{Top}}(Y, Z) = \mathrm{Map}(Y, Z)$. We have

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$
$$\mathrm{Map}_G(X \times Y, Z) \cong \mathrm{Map}_G(X, \mathrm{Map}(Y, Z)).$$

All the terminologies above have based version:

Definition 1.4. A (left) based G-space is a G-space X with a G-fixed basepoint. A based G-map $f : X \to Y$ is a G-map f that sends the basepoint of X to the basepoint of Y. We use $\mathrm{Map}_{G,*}(X, Y)$ to denote the space of all based G-maps.

We use $G\mathrm{Top}_*$ to denote the category of based G-spaces and based G-maps. We have

$$\mathrm{Map}_*(X \wedge Y, Z) \cong \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$$
$$\mathrm{Map}_{G,*}(X \wedge Y, Z) \cong \mathrm{Map}_{G,*}(X, \mathrm{Map}_*(Y, Z)).$$

They together form the category of based G-spaces, $G\mathrm{Top}_*$.

There is a functor $G\mathrm{Top} \to G\mathrm{Top}_*, X \mapsto X_+$ which is disjoint union with an addition point $*$ with trivial G-action. It is the left adjoint to the forgetful functor $G\mathrm{Top}_* \to G\mathrm{Top}$.

Definition 1.5. A G-CW complex X is the union of sub G-spaces X^n such that X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching G-cells $G/H \times D^{n+1}$ along the attaching G-maps $G/H \times S^n \to X^n$.

If we recall the definition of CW complex introduced yesterday, compare the pushout diagram.

$$\begin{array}{ccc} \coprod_\alpha S^n & \longrightarrow & X^n \\ \downarrow & & \downarrow \\ \coprod_\alpha D^{n+1} & \longrightarrow & X^{n+1} \end{array}$$

Remark: Compare the pushout diagrams of non-equivariant CW complex and of equivariant CW complex. To give a sense that G/H plays the role of points in nonequivariant case.

$$\begin{array}{ccc} \coprod_\alpha G/H_\alpha \times S^n & \longrightarrow & X^n \\ \downarrow & & \downarrow \\ \coprod_\alpha G/H_\alpha \times D^{n+1} & \longrightarrow & X^{n+1} \end{array}$$

The attaching map $G/H \times S^n \to X^n$ is determined by its restriction $S^n \to (X^n)^H$.

In equivariant homotopy theory, orbits G/H play the role of points.

Example 1.6. Use the examples of S^1 with reflection action of $\mathbb{Z}/2$ and S^1 with antipodal action of $\mathbb{Z}/2$.

2 Cellular theory

Definition 2.1. A G-homotopy of $f, g\colon X \to Y$ is a G-map $H : I \times X \to Y$ with G acting trivially on $I = [0, 1]$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$.

We use $[X, Y]_G$ to denote the homotopy classes of G-maps.

Definition 2.2. For a topological G-space X, $H \subset G$ a closed subgroup of G, its nth H-equivariant homotopy groups are

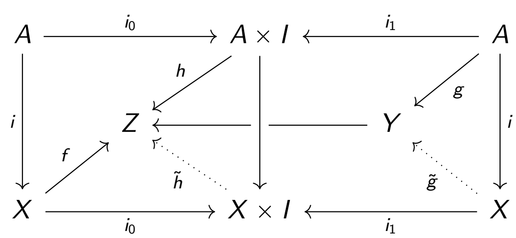
$$\pi_n^H(X) = \pi_0 \mathrm{Hom}_G(G/H_+ \wedge S^n, X) = \pi_n(X^H).$$

Definition 2.3. A G-map $f : X \to Y$ is weak (homotopy) equivalence if $f^H : X^H \to Y^H$ is a weak equivalence for all $H \subset G$.

Recall that in non-equivariant case, a map $f : Y \to Z$ between non-empty spaces is an n-equivalence for $n \geqslant 0$ if $\pi_p(f)$ is a bijection for $q < n$ and a surjection for $q = n$ (for any choice of basepoint). To accommodate empty spaces, we say $f : \varnothing \to Z$ is a (-1)-equivalence when $Z = \varnothing$, and f is not an n-equivalence for any n when $Z \neq \varnothing$. Now we give a analogous definition for equivariant case.

Definition 2.4. Let ν be a function from conjugacy classes of subgroups of G to the integers $\geqslant -1$. We say that a map $e : Y \to Z$ is a ν-equivalence if $e^H : Y^H \to Z^H$ is a $\nu(H)$-equivalence for all H. We say a G-CW complex X has dimension $\leqslant \nu$ if its cells of orbit type G/H have dimensions $\leqslant \nu(H)$.

Theorem 2.5 (Homotopy extension and lifting property)**.** *Let A be a subcomplex of a G-CW complex X of dimension $\leqslant \nu$ and let $e : Y \to Z$ be a ν-equivalence. Suppose given maps $g : A \to Y$, $h : A \times I \to Z$, and $f : X \to Z$ such that $eg = hi_1$ and $fi = hi_0$ in the following diagram: then there exists maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commutes.*



Proof. We construct $\tilde{g}$ and $\tilde{h}$ on $A \cup X^n$ by induction on n. Passing from the n-skeleton to the $(n+1)$-skeleton, we may work one cell of X not in A at a time. By considering attaching mas, we quickly reduce the proof to the case when $(X, A) = (G/H \times D^{n+1}, G/H \times S^n)$ and this reduces to the nonequivariant case of (D^{n+1}, S^n) □

Theorem 2.6 (Whitehead Theorem)**.** *Let $e : Y \to Z$ be a ν-equivalence and X be a G-CW complex. Then $e_* : [X, Y]_G \to [X, Z]_G$ is a bijection if X has dimension less than ν and a surjection if X has dimension ν.*

Proof. Apply Theorem 2.5 to $(X, \varnothing)$ for the surjectivity and to $(X \times I, X \times \partial I)$ for the injectivity. □

Corollary 2.7. *If $e : Y \to Z$ is a ν-equivalence between G-CW complexes of dimension less than ν, then e is a G-homotopy equivalence.*

We also have the equivariant cellular approximation theorem (skipped) and the G-CW approximation theorem

Theorem 2.8. *For any G-space X, there is a G-CW complex ΓX and a weak equivalence $\Gamma X \to X$.*

The equivariant Whitehead theorem implies that ΓX is unique up to G-homotopy equivalence. It is possible to construct Γ functorialy [Seymour].

3 Fixed points and orbits

Let X be a G-space. For a closed subgroup $H \subset G$, the H-fixed points of X is

$$X^H = \{x | hx = x \text{ for all } h \in H\};$$

the H-orbits of X is

$$X_H = X/H = X/(x \sim hx \text{ for } x \in X \text{ and } h \in H).$$

The Weyl group is

$$W_G H = N_G H/H \cong \mathrm{Hom}_G(G/H, G/H).$$

Both X^H and X_H are $W_G H$-spaces. The $W_G H$-action on X^H can be seen using $X^H \cong \mathrm{Hom}_G(G/H, X)$, and the $W_G H$-action on X_H can be checked by hand.

Consider the functor $F = (-)^{\mathrm{triv}} : \mathrm{Top} \to G\mathrm{Top}$ that sends a space Y to Y with trivial G-action. It has both right and left adjoints, which are fixed points and orbits.

$$G\mathrm{Top}(X^{\mathrm{triv}}, Y) \cong \mathrm{Top}(X, Y^G)$$
$$G\mathrm{Top}(X, Y^{\mathrm{triv}}) \cong \mathrm{Top}(X_G, Y)$$

In general, if we have a map of groups $f : H \to K$, it induces a functor

$$f^* : K\mathrm{Top} \to H\mathrm{Top}.$$

The left adjoint of f^* is

$$f_!(X) = K \times_H X$$

and the right adjoint of f^* is

$$f_*(X) = \mathrm{Map}(K, X)^H.$$

We spell out two important cases.

1. For $f : G \to \{e\}$, $f^* = (-)^{\mathrm{triv}}$ and we recover the adjunctions above.
2. For $f = i : H \hookrightarrow G$ being an inclusion of subgroup, i^* is the restriction of the G-action to the H-action, and we have

$$H\mathrm{Top}(i^*X, Y) \cong G\mathrm{Top}(X, \mathrm{Map}_H(G, Y));$$
$$H\mathrm{Top}(X, i^*Y) \cong G\mathrm{Top}(G \times_H X, Y)$$

We also have the based version

$$H\mathrm{Top}_*(i^*X, Y) \cong G\mathrm{Top}_*(X, \mathrm{Map}_{H,*}(G_+, Y));$$
$$H\mathrm{Top}_*(X, i^*Y) \cong G\mathrm{Top}_*(G_+ \wedge_H X, Y).$$

Exercise 3.1. Verify that the attaching map $G/H \times S^n \to X^n$ is determined by its restriction $S^n \to (X^n)^H$.

The fixed points and orbits are limits and colimits. Let BG be the category of one object with G being automorphism monoid of this object. Then we have a (naive) identification

$$G\mathsf{Top} \cong \mathsf{Fun}(BG, \mathrm{Top}). \tag{1}$$

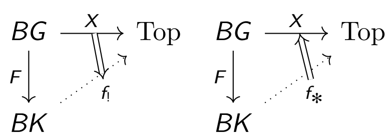
For a G-space X regarded as a functor, we have

$$X^G = \lim_{BG} X.$$

$$X_G = \operatorname{colim}_{BG} X.$$

To see this, note that we have canonical maps $X^G \to X$ and $X \to X_G$ and one can check that they satisfy the universal properties for limits and colimits.

In general, a map of groups $f : G \to K$ induces a functor $F : BG \to BK$. Recall that we have the functor $f^* : K\mathrm{Top} \to G\mathrm{Top}$ with left adjoint $f_!$ and right adjoint f_*. Under the identifying Equation 1, $f_!$ is the left Kan extension along F and f_* is the right Kan extension along F.



The diagrams do not commute. There are natural transformation $\mathrm{id} \Rightarrow f^* f_!$ and $f^* f_* \Rightarrow \mathrm{id}$. In particular, if we take $K = \{e\}$, left and right Kan extensions of a functor X along F to the trivial category give the colimits and limits of X. Thus, we recover the special case discussed above.

4 Homotopy fixed points and orbits

Now we define homotopy fixed points of a G-space X. Recall that EG is a (right) G-space such that the G-action is free and that EG is non-equivariantly contractible.

Definition 4.1. The homotopy G-fixed point space of X is

$$X^{hG} = \mathsf{Map}_G(EG, X) = \mathsf{Map}(EG, X)^G.$$

The homotopy G-orbit space of X is

$$X_{hG} = EG \times_G X = (EG \times X)_G.$$

Exercise 4.2. Find the homotopy fixed point and homotopy orbit of a space X with the trivial G-action.

Remark 4.3. In $G\mathrm{Top}_*$, the homotopy fixed points and orbits are $\mathrm{Map}(EG_+, X)^G$ and $EG_+ \wedge_G X$.

The word homotopy is justified by the fact that

$$X^{hG} \simeq \mathrm{holim}_{BG} X;$$
$$X_{hG} \simeq \mathrm{hocolim}_{BG} X.$$

We first define the homotopy colimit of a diagram $X : \mathcal{D} \to \mathrm{Top}$ for a small topological category $\mathcal{D}$. The simplical replacement of X, called the bar construction, is the simplicial space

$$B_n(*, \mathcal{D}, X) = \{(\underline{f}, s) | \underline{f} = (f_1 : d_1 \to d_0, \cdots, f_n : d_n \to d_{n-1}), s \in X(d_n)\},$$

where $\underline{f}$ consists of n composable arrows in $\mathcal{D}$. The face maps except for the first and last one compose the morphisms in $\mathcal{D}$, the first face map forgets f_1, and the last face map forgets f_n and replaces s by $(f_n)_*(s) \in X(d_{n-1})$. The degeneracy maps insert identity morphisms in $\mathcal{D}$.

Definition 4.4. The homotopy colimit $\mathrm{hocolim}_{\mathcal{D}} X$ is the geometric realization $|B_\bullet(*, \mathcal{D}, X)|$.

We specialize to $\mathcal{D} = BG$ and write $\mathcal{D} = G$ in the bar construction. (This difference of notation is due to difference of authors.) Dropping the constant contravariant functor $*$ from the notation, we have $B_\bullet(G, X) \cong B_\bullet(*, G, G) \times_G X$ and so $|B_\bullet(G, X)| \cong |B_\bullet(*, G, G)| \times_G X \cong EG \times_G X$. (In the second bar construction, the first G is the category BG; the second G is the functor $BG \to \mathrm{Top}$ corresponding to the left G-space G.) Notice that $B_1(G, X) = G \times X$, $B_0(G, X) = X$, the face map d_0 projects to X and d_1 is the G-action map. We have a map $X_{hG} \to X_G$ defined by

$$\mathrm{hocolim}_{BG} X = |B_\bullet(G, X)| \to \mathrm{coeq}(d_0, d_1 : G \times X \to X) = \mathrm{colim}_{BG} X$$

Dually, we have the cobar construction, a cosimplicial space $C_\bullet(G, X)$ and $\mathrm{holim}_{BG}(X) = Tot(C_\bullet(G, X))$. Explicitly, we have

$$C_n(G; X) = \prod_{\underline{f} = (f_1 : d_1 \to d_0, \cdots, f_n : d_n \to d_{n-1})} X(d_0).$$

We have $C_1(G, X) = \mathrm{Map}(G, X)$, $C_0(G, X) = X$, and for $x \in X$, the coface map $d^0(x)$ sends g to gx; $d^1(x)$ is the constant map $G \to X$ at x. The map $X^G \to X^{hG}$ given by

$$\lim_{BG} X = \mathrm{eq}(d^0, d^1 : X \to \mathrm{Map}(G, X)) \to |C_\bullet(G, X)| \simeq \mathrm{holim}_{BG} X.$$

I must admit that the dual constructions are mathematically identical but conceptually harder to internalize.

Remark 4.5. The homotopy limits and colimits are attempts to homotopical replace the limit and colimit functors. Using model category techniques, we have the following formulation. The colimit and limit functors are left and right adjoints to the constant diagram functor $sSet \to \mathrm{Fun}(\mathcal{D}, sSet)$. They are Quillen adjoints if we endow the functor category with the suitable projective or injective model structure. It turns out ([BK, XI.8.1,XII.2.4])that the constant diagram functor on the homotopy category $Ho(sSet) \to \mathrm{Ho}(\mathrm{Fun}(\mathcal{D}, sSet))$ has left adjoint

$$\mathbb{L}\mathrm{hocolim}_{\mathcal{D}} : \mathrm{Ho}(\mathrm{Fun}(\mathcal{D}, sSet)) \to \mathrm{Ho}(sSet)$$

and right adjoint

$$\mathbb{R}\mathrm{holim}_{\mathcal{D}} : \mathrm{Ho}(\mathrm{Fun}(\mathcal{D}, sSet)) \to \mathrm{Ho}(sSet).$$

So using the fibrant replacement functor P_{inj} and the cofibrant replacement functor Q_{proj} for the corresponding model structures, we have for levelwise fibrant/cofibrant functors F,

$$\mathrm{holim} F \simeq \mathbb{R}\mathrm{lim}(F) \simeq \mathrm{lim}(P_{inj} F)$$
$$\mathrm{hocolim} F \simeq \mathbb{L}\mathrm{colim}(F) \simeq \mathrm{colim}(Q_{proj} F)$$

The maps

$$X^G \to X^{hG} \text{ and } X_{hG} \to X_G$$

can be seen concretely using Definition 4.1. We have a map $EG \to *$. This map induces $X = \mathrm{Map}(*, X) \to \mathrm{Map}(EG, X)$. Taking G-fixed points, it gives $X^G \to X^{hG}$. It also induces $EG \times X \to * \times X = X$. Taking the G-orbits, it gives $X_{hG} \to X_G$. Note that the map $EG \to *$ is a nonequivariantly homotopy equivalence, but not a G-homotopy equivalence. Therefore, the maps $X^G \to X^{hG}$ and $X_{hG} \to X_G$ are in general not homotopy equivalences.

For a diagram $F : I \to \mathrm{Top}$, the universal property of the colimit and limit of F can be written as

$$\mathrm{Map}(\mathrm{colim}_I F, X) \cong \mathrm{lim}_I \mathrm{Map}(F(i), X)$$
$$\mathrm{Map}(X, \mathrm{lim}_I F) \cong \mathrm{lim}_I \mathrm{Map}(X, F(i))$$

For the homotopy versions, we also have natural weak equivalences of spaces

$$\mathrm{Map}(\mathrm{hocolim}_I F, X) \simeq \mathrm{holim}_I \mathrm{Map}(F(i), X)$$
$$\mathrm{Map}(X, \mathrm{holim}_I F) \simeq \mathrm{holim}_I \mathrm{Map}(X, F(i))$$

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