Equivariant spaces

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Plan
1. G-spaces and G-CW complexes
2. Equivariant cellular theory
3. Fixed points and orbits
4. Homotopy fixed points and orbits

1. G-spaces and G-CW complexes

We take all spaces to be compactly generated and weak Hausdorff.

G: topological group, compact Lie group, finite group

Definition
- left G-space: space X with G continuous map $G\times X \to X$, $(g,x)\mapsto gx$ such that $g_1(g_2 x) = (g_1 g_2)x$, $ex = x$
- G-map (equivariant map): $f: X \to Y$ such that $f(gx) = g f(x)$

Remark: right G-space: X: right G-space, then it can be regarded as a left G-space by $gx = xg^{-1}$.

GTop and Top$_G$: obj: G-spaces

mor: $G\mathrm{Top}(X,Y) = \mathrm{Map}_G(X,Y)$ G-maps, a space

$\mathrm{Top}_G(X,Y) = \mathrm{Map}(X,Y)$, a G-space. G acts by conjugation

$$\mathrm{Map}_G(X,Y) = (\mathrm{Map}(X,Y))^G$$

$f: X\to Y$
$\Rightarrow (gf): X \to Y$
$(gf)(x) = gf(g^{-1}x) \in Y$

$(G\mathrm{Top}, \times, \mathrm{pt})$ is a closed Cartesian monoidal category

Cartesian product. $X, Y \in G\mathrm{Top}$. $X\times Y \in G\mathrm{Top}$, G acts diagonally $g(x,y) = (gx, gy)$

internal hom $G\mathrm{Top}(X,Y) = \mathrm{Map}(X,Y) = \mathrm{Top}_G(X,Y)$ is a G-space

- $\mathrm{Map}_G(X\times Y, Z) \cong \mathrm{Map}_G(X, \mathrm{Map}(Y,Z))$

obtained from taking G-f.p. of

- $\mathrm{Map}(X\times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y,Z))$

The pointed version

$G\mathrm{Top}_*$ obj: based G-spaces. X: G-space $* \in X$ such that $*$ is G-fixed.

The pointed version

$GTop_*$ obj : based G-spaces. X : G-space $* \in X$ such that $*$ is G-fixed.

mor : based G-maps (is a based space, based at the constant map to $*$.)

- $Map_*(X \wedge Y, Z) \cong Map_*(X, \underline{Map_*(Y, Z)})$

  $Map_{G*}(X \wedge Y, Z) \cong Map_{G*}(X, Map_*(Y, Z))$

Adjunction

$$\begin{tikzcd} GTop \arrow[r, shift left, "( )_+"] \arrow[r, phantom, "\perp"] & GTop_* \arrow[l, shift left, "fgt"] \end{tikzcd} \qquad GTop_*(X_+, Y) \cong GTop(X, Y)$$

Definition. G-CW complex $X = \bigcup_n X^n$

$X^0 = \coprod_i G/H_i$

attaching map (G-map)

$$\begin{tikzcd} \coprod_\alpha G/H_\alpha \times S^n \arrow[r] \arrow[d, "\text{inclusion of } \partial"'] & X^n \arrow[d] \\ \coprod_\alpha G/H_\alpha \times D^{n+1} \arrow[r] & X^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The attaching map $G/H \times S^n \to X^n$ is determined by its restriction $S^n \to (X^n)^H$.

Remark: orbits G/H play the role of points.

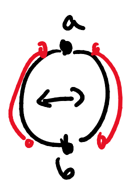
orbit category O_G

obj : G/H for closed subgroups $H \subseteq G$

mor: G-maps

Example.

- The circle with the reflection action.

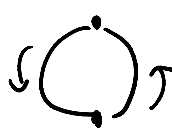


$G = C_2$

0-cells $\underset{a}{\underset{\shortparallel}{C_2/C_2}}$ and $\underset{b}{\underset{\shortparallel}{C_2/C_2}}$

1-cell $C_2/e \times D^1$

- The circle with the rotation action.



$G = C_2$

0-cell C_2/e

1-cell $C_2/e \times D^1$

- EG and BG

  EG : free G-space such that $EG \simeq *$.

  $BG = EG/G$ space

examples. $BO(n)$ Grassmannian

$EO(n)$ framings in Grassmanian

2. Equivariant Cellular Theory

Definition. G-homotopy

$f \simeq g : X \to Y$
such that there is $H : X \times I \to Y$ (G-map)
$f = H(-, 0)$, $I = [0,1]$ with trivial G-action
$g = H(-, 1)$,

We use $[X, Y]_G$ to denote the homotopy classes of G-maps.

Definition. H-equivariant homotopy groups

$H \subseteq G$ closed subgroup

$$\pi_n^H(X) = \pi_0 \underline{\mathrm{Hom}_G(G/H_+ \wedge S^n, X)} = \boxed{\pi_n(X^H).}$$

Definition. Weak (homotopy) equivalence

$f : X \to Y$ is a w.e.
if $\forall H$, $f^H : X^H \to Y^H$ is a w.e.

homotopy category
$\mathcal{C}$, $\mathcal{W}$ morphisms in $\mathcal{C}$
w.e.
$\Rightarrow Ho(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$

Remark. Borel weak equivalence

$f : X \to Y$ is a Borel w.e.
if $f^e : X^e \to Y^e$ (underlying map)
is a w.e.

Definition. n-equivalence (non-equivariant)

$f : X \to Y$
$(n \geq 0)$ if $\pi_q(f)$ is a bijection for $q < n$ and a surjection for $q = n$.
$(n = -1)$ $f : \phi \to Y$ is a (-1)-equi $\Leftrightarrow$ $Y = \phi$

Definition 2.4. Let ν be a function from conjugacy classes of subgroups of G to the integers $\geqslant -1$. We say that a map $e : Y \to Z$ is a ν-equivalence if $e^H : Y^H \to Z^H$ is a $\nu(H)$-equivalence for all H. We say a G-CW complex X has dimension $\leqslant \nu$ if its cells of orbit type G/H have dimensions $\leqslant \nu(H)$.

HELP

$A \hookrightarrow X$ $\dim X \leq \nu$

Theorem 2.5 (Homotopy extension and lifting property). *Let A be a subcomplex of a G-CW complex X of dimension $\leqslant \nu$ and let $\underline{e : Y \to Z}$ be a ν-equivalence. Suppose given maps $g : A \to Y$, $h : A \times I \to Z$, and $f : X \to Z$ such that $eg = hi_1$ and $fi = hi_0$ in the following diagram: then there exists maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commutes.*

$$A \xrightarrow{i_0} A \times I \xleftarrow{i_1} A$$

$$A \xrightarrow{g} Y$$

$$
\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[dl, "h"'] \arrow[d] & & A \arrow[ll, "i_1"'] \arrow[d, "i"] \arrow[dl, "g"'] \\
X \arrow[r, "i_0"'] \arrow[ur, phantom] & X \times I & Y \arrow[l, phantom] & X \arrow[ll, "i_1"] \arrow[ul, dotted, "\tilde{g}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[rrr, "i_0"] \arrow[ddd, "i"'] & & & A \times I \arrow[ddd] \arrow[dll, "h"'] & & & A \arrow[lll, "i_1"'] \arrow[ddd, "i"] \arrow[dl, "g"'] \\
 & Z & & & & Y \arrow[llll, "e"'] & \\
 & & & & & & \\
X \arrow[rrr, "i_0"'] \arrow[uur, "f"] & & & X \times I \arrow[uull, dotted, "\tilde{h}"'] & & & X \arrow[lll, "i_1"] \arrow[uul, dotted, "\tilde{g}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[d, hook] & {} \arrow[d, "e"] \arrow[dl, "h"'] \\
X \arrow[r, "f"'] \arrow[ur, "\tilde{g}"] & Z
\end{tikzcd}
$$

① $\tilde{g}$ lifts g

② $$
\begin{tikzcd}
X\times\{0\} \cup A \times I \cup X\times\{1\} \arrow[r, "f \cup h \cup e\tilde{g}"] \arrow[d, hook] & Z \\
X\times I \arrow[ur] &
\end{tikzcd}
$$

- Theorem (equivariant Whitehead theorem) $e : Y \to Z$ ν-equi, X : G-CW

  $e_* : [X, Y]_G \to [X, Z]_G$ is a $\begin{cases} \text{bijection} & \text{if } \dim X < \nu \\ \text{surjection} & \text{if } \dim X \leq \nu \end{cases}$

PROOF. Apply HELP to the pair $(X, \emptyset)$ for the surjectivity. Apply HELP to the pair $(X \times I, X \times \partial I)$ for the injectivity. $\square$

- Corollary $e : Y \to Z$ ν-equi, Y, Z : G-CW, $\dim < \nu$

  $\Rightarrow$ e is a G-homotopy equivalence.

- Theorem (G-CW approximation) $\boxed{\text{G-CW}/\text{h.e} \simeq \mathrm{Ho}(G\mathrm{Top})}$

  For any G-space X, there is a G-CW complex ΓX and a w.e. $\Gamma X \to X$.

  The Γ can be taken functorially.

3. Fixed points and orbits

X : G-space $H \subseteq G$

$$
\left.
\begin{aligned}
&X^H = \{x \mid hx = x \text{ for all } h \in H\}; \\
&X^H \cong \mathrm{Map}_G(G/H, X) \\
&X_H = X/H = X/(x \sim hx \text{ for } x \in X \text{ and } h \in H).
\end{aligned}
\right\} W_G H\text{-spaces}
$$

- $W_G H = N_G H / H \cong \mathrm{Hom}_G(G/H, G/H)$.

  - As adjoints

$$
\begin{tikzcd}
(-)^{\mathrm{triv}} : \mathrm{Top} \arrow[r, "\perp"] & G\mathrm{Top} \arrow[l, bend right, "(\ )_G"'] \arrow[l, bend left, "(\ )^G", "\perp"']
\end{tikzcd}
$$

$$G\mathrm{Top}(X^{\mathrm{triv}}, Y) \cong \mathrm{Top}(X, Y^G)$$

$$G\mathrm{Top}(X, Y^{\mathrm{triv}}) \cong \mathrm{Top}(X/G, Y)$$

- The general formulation and two special cases

$f: H \to K$ group homomorphism

$$\begin{tikzcd} K\mathrm{Top} \arrow[r, "f^*"] & H\mathrm{Top} \arrow[l, bend right=50, "f_!"'] \arrow[l, bend left=50, "f_*"] \end{tikzcd}$$

$$f_!(X) = K \underset{H}{\times} X$$

$$f_*(X) = \mathrm{Map}(K, X)^H$$

① $K = \{e\}$. $\quad f^* = (-)^{\mathrm{triv}}$.

$f: G \to \{e\}$

$$\Rightarrow f_!(X) = * \underset{G}{\times} X = X/G$$

$$f_*(X) = \mathrm{Map}(*, X)^G = X^G$$

② $f: H \hookrightarrow G \qquad f^* = \mathrm{Res}^G_H$

$$\Rightarrow f_!(X) = G \underset{H}{\times} X$$

$$f_*(X) = \mathrm{Map}_H(G, X)$$

Exercise 3.1. Verify that the attaching map $G/H \times S^n \to X^n$ is determined by its restriction $S^n \to (X^n)^H$.

- As limits and colimits $\quad BG$: category of one obj $* \circlearrowleft G$
(some authors call this category G)

$$G\mathrm{Top} \cong \mathrm{Fun}(BG, \mathrm{Top}) \qquad \text{naive identification}$$

$X \in G\mathrm{Top}$ $\qquad \mathcal{X}: G \circlearrowleft \longmapsto X \circlearrowleft$

To see this, we have

$$\boxed{\begin{aligned} X^G &= \lim_{BG} \mathcal{X} \\ X_G &= \operatorname{colim}_{BG} \mathcal{X} \end{aligned}}$$

inclusion of f.p. $\quad X^G \to X \circlearrowleft G$

projection to orbits $\quad {}_G X \to X_G$ (G)

- Aside: The general formulation

$$f: G \to K \Rightarrow F: BG \to BK$$

$$\Rightarrow \quad \begin{array}{ccc} \mathrm{Fun}(BK, \mathrm{Top}) & \xrightarrow{F^*} & \mathrm{Fun}(BG, \mathrm{Top}) \\ \cong & & \cong \\ K\mathrm{Top} & \xrightarrow{f^*} & G\mathrm{Top} \end{array}$$

$\Rightarrow$ F^* has left and right adjoints.

$$\begin{tikzcd} BG \arrow[r, "X"] \arrow[d, "F"'] & \mathrm{Top} \\ BK \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{array}{l} \mathrm{Lan}_F X = f_! X \\ \mathrm{Ran}_F X = f_* X \end{array}$$

When $K = \{e\}$, $BK = *$, $F: BG \to *$.

$$\mathrm{Lan}_F X = \operatorname{colim}_{BG} X,$$

$$\mathrm{Ran}_F X = \lim_{BG} X.$$

4. Homotopy fixed points and homotopy orbits

EG: free G-space such that $EG \simeq *$

Definition 4.1. The homotopy G-fixed point space of X is

$$X^{hG} = \mathrm{Map}_G(EG, X) = \mathrm{Map}(EG, X)^G.$$

The homotopy G-orbit space of X is

$$X_{hG} = EG \times_G X = (EG \times X)_G.$$

In $G\mathrm{Top}_*$

$$X^{hG} = \mathrm{Map}_{G,*}(EG_+, X)$$

$$X_{hG} = EG_+ \wedge_G X$$

Exercise 4.2. Find the homotopy fixed point and homotopy orbit of a space X with the trivial G-action.

$$X^{hG} = \mathrm{Map}(BG, X) \not\simeq X^G$$

$$X_{hG} = BG \times X \not\simeq X_G$$

- Goal: $X^{hG} \simeq \mathrm{holim}_{BG} X$; $X_{hG} \simeq \mathrm{hocolim}_{BG} X$.

  We have natural maps $X^G \to X^{hG}$, $X_{hG} \to X_G$

- Why do we study homotopy (co)limits of diagrams?

$$\mathcal{D} = \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \end{tikzcd} \qquad F_1 \quad \begin{tikzcd} S^0 \arrow[r] \arrow[d] & {[0,1]} \arrow[d] \\ * \arrow[r, hook] & S^1 \end{tikzcd} \qquad F_2 \quad \begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r, hook] & * \end{tikzcd}$$

$F_1 \to F_2$ levelwise w.e.

$\Rightarrow$ $\operatorname{colim} F_1 \to \operatorname{colim} F_2$ not a w.e.

- Homotopy invariance $F_1 \to F_2$ levelwise w.e.

Prop. ① If the image of the objects in F_1, F_2 are all cofibrant

then $\operatorname{hocolim} F_1 \longrightarrow \operatorname{hocolim} F_2$ is a w.e.

② If $\cdots\cdots$ are all fibrant (automatic in Top)

then $\operatorname{holim} F_1 \to \operatorname{holim} F_2$ is a w.e.

- Definition, the (co)simplicial one

$\mathcal{D}$: topogical category (small) $X: \mathcal{D} \to Top$.

$$B_n(*, \mathcal{D}, X) = \{(\underline{f}, s) \mid \underline{f} = \underbrace{i_0 \xleftarrow{f_1} i_1 \leftarrow \cdots \xleftarrow{f_n} i_n}_{n \text{ composable morphisms in } \mathcal{D}},\ s \in X(i_n)\}$$

$B_n(*, \mathcal{D}, X) \to B_{n-1}(*, \mathcal{D}, X)$

face map d_0 forgets f_1

d_i, $1 \le i \le n-1$ compose $f_{i-1} f_i$

d_n forgets f_n, replace s by $(f_n)_*(s) \in X(i_{n-1})$

degeneracy map inserts $i_k \xleftarrow{id} i_k$.

$\Rightarrow$ Def. $\operatorname{hocolim}_{\mathcal{D}} X = |B_*(*, \mathcal{D}, X)|$.

Example. $\mathcal{D} = BG$. we write $\mathcal{D} = G$ in the bar construction.

$$B_*(*, G, *) \qquad \cdots\ \underset{B_2}{G \times G} \substack{\to\\\to\\\to} \underset{B_1}{G} \substack{\to\\\to} \underset{B_0}{*}$$

$\Rightarrow |B(*, G, *)| \simeq BG$

$d_0(g_1, g_2) = g_2$

$d_1(g_1, g_2) = g_1 g_2$

$d_2(g_1, g_2) = g_1$

$$B_*(*, G, G) \qquad \cdots\ G \times G \times G \substack{\to\\\to\\\to} G \times G \substack{\to\\\to} G$$

$\Rightarrow |B(*, G, G)| \simeq EG$

$d_0(g_0, g_1) = g_1$

$d_1(g_0, g_1) = g_0 g_1$

$B_*(*, G, X) \qquad \cdots \;\; \substack{\to\\\to\\\to} \;\; \boxed{G \times X \rightrightarrows X}$

$d_0(g,x) = x$

$d_1(g,x) = gx$

$\Rightarrow |B_*(*, G, X)| = |B_*(*, G, G) \times_G X|$

$\cong |B_*(*, G, G)| \times_G X \simeq EG \times_G X$

$\Rightarrow \quad |B_*(*, G, X)| \longrightarrow \mathrm{coeq}(G \times X \rightrightarrows X) = X_G$

$\Rightarrow \quad X_{hG} \to X_G$

Or, using $EG \to *$, we have $EG \times X \to * \times X = X$

$\Rightarrow \quad \underset{\substack{\parallel \\ X_{hG}}}{EG \times_G X} \to X_G$

Dually, we have a cosimplicial space, $C_*(G, X)$ called cobar construction.

$X^{hG} = \operatorname{holim}_{BG} X = \mathrm{Tot}(C_*(G, X))$

and we have $\quad X^G \to X^{hG}$

- As adjunctions

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "\mathrm{const}"] & \mathrm{Top}^{\mathcal{D}} \arrow[l, bend right, "\mathrm{colim}"', "\perp"] \arrow[l, bend left, "\mathrm{lim}", "\perp"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Top} \arrow[r, bend right, "\mathrm{Map}(B(-/\mathcal{D}), -)"'] & \mathrm{Top}^{\mathcal{D}} \arrow[l, bend right, "\mathrm{hocolim}"', "\perp"] \end{tikzcd}$$

$X \longmapsto \mathcal{D} \to \mathrm{Top}$

$\boxed{\text{example}}$

$\mathcal{D} = BG$.

$*/BG \cong \widetilde{G}$

obj : G

mor : $g_1 \xrightarrow{g} gg_1$

$B(*/BG) \simeq EG$

$\mathrm{Map}(B(-/\mathcal{D}), -)$

$$X \longmapsto \begin{array}{l} \mathcal{D} \to \mathrm{Top} \\ d \mapsto \mathrm{Map}(B(d/\mathcal{D}), X) \\ d_1 \to d_2 \quad d_1/\mathcal{D} \leftarrow d_2/\mathcal{D} \end{array}$$

$B(*/BG) \subseteq EG$

$$\mathrm{Top} \underset{\mathrm{Map}(EG,-)}{\overset{(\ )^{hG}}{\rightleftarrows}} \mathrm{Top}^{BG}$$

$$\mathrm{Top} \underset{\mathrm{holim}}{\overset{-\times B(\mathcal{D}/-)}{\rightleftarrows}} \mathrm{Top}^{\mathcal{D}}$$

$$\mathrm{Top} \underset{(\ )^{hG}}{\overset{-\times EG}{\rightleftarrows}} \mathrm{Top}^{BG}$$

- (Aside) using model category

$\mathrm{Top}^{\mathcal{D}}$ has injective model structure levelwise cof, w.e
& projective model structure levelwise fib, w.e.

① $$\mathrm{Top} \underset{\mathrm{const}}{\overset{\mathrm{colim}}{\leftrightarrows}} (\mathrm{Top}^{\mathcal{D}})_{proj}$$

$$\Rightarrow \quad \mathrm{Ho}(\mathrm{Top}) \underset{\mathrm{const}}{\overset{L}{\leftrightarrows}} \mathrm{Ho}(\mathrm{Top}^{\mathcal{D}}) \qquad LX \simeq \mathrm{colim}(\underbrace{Q_{proj}X}_{\text{hard}})$$

② $$\mathrm{Top} \underset{\mathrm{Map}(B(-/\mathcal{D}),-)}{\overset{\mathrm{hocolim}}{\leftrightarrows}} (\mathrm{Top}^{\mathcal{D}})_{inj}$$

$$\Rightarrow \quad \mathrm{Ho}(\mathrm{Top}) \underset{\mathrm{const}}{\overset{L}{\leftrightarrows}} \mathrm{Ho}(\mathrm{Top}^{\mathcal{D}}) \qquad LX \simeq \mathrm{hocolim}(Q_{inj}X)$$

$\simeq \mathrm{hocolim}\, X$

if $X(d)$ are all cofibrant for $d \in \mathcal{D}$

- Universal properties

$$\mathrm{Map}(\mathrm{colim}_I F, X) \cong \lim_I \mathrm{Map}(F(i), X)$$
$$\mathrm{Map}(X, \lim_I F) \cong \lim_I \mathrm{Map}(X, F(i))$$

$$\mathrm{Map}(\mathrm{hocolim}_I F, X) \simeq \mathrm{holim}_I \mathrm{Map}(F(i), X)$$
$$\mathrm{Map}(X, \mathrm{holim}_I F) \simeq \mathrm{holim}_I \mathrm{Map}(X, F(i))$$

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & F_3 \arrow[d] \arrow[ddr, bend left] & \\ F_2 \arrow[r] \arrow[drr, bend right] & \mathrm{colim}F \arrow[dr] & \\ & & X \end{tikzcd}$$

$$\mathrm{Map}(\mathrm{hocolim}_I F, X) \simeq \mathrm{holim}_I \mathrm{Map}(F(i), X)$$
$$\mathrm{Map}(X, \mathrm{holim}_I F) \simeq \mathrm{holim}_I \mathrm{Map}(X, F(i))$$

X

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] & F_3 \arrow[d] \arrow[ddr, bend left] & \\
F_2 \arrow[r] \arrow[drr, bend right] & F_2 \cup F_1 \times I \cup F_3 \arrow[dr, dashed] & \\
 & & X
\end{tikzcd}$$

not unique, but has a contractible space of choices