Talk 13
Bredon homology and Smith theory

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Recall: Elmendorf theorem

Thm (Elmendorf)

There is a Quillen equivalence between $\mathscr{P}(\mathrm{Orb}_G)$ and GTop.

$GTop \to P(\mathrm{Orb}_G)$: taking fixed points

- Upshot: Fixed points contains enough information.
- "Everything is over Orb_G":

Example $\quad \pi_*^H(-X) := \pi_n(X^H)$

$\rightsquigarrow$ assembles into $\underline{\pi}_*(X) : \mathrm{Orb}_G \to Ab$

Coefficient system

- Definition: Orbit category
  - objects: G/H orbits
  - morphisms: $G/H \to G/K$ iff $H \subseteq rKr^{-1}$ $\exists r \in G$
  - an example: $G = C_2$ $\quad \mathrm{id} \circlearrowright C_2/e \xrightarrow{\text{proj}} C_2/C_2 \circlearrowleft \mathrm{id}$

    $\circlearrowleft$ swith
- Definition: Coefficient system

  A coefficient system is a functor $\mathrm{Orb}_G^{op} \to Ab$.
- The category of coefficient system is an Abelicen cat.
- Examples
  - constant coefficient system $\quad B \in Ab$

    $\underline{B}(G/H) = B$

    $\underline{B}(f) = \mathrm{id}_B$
  - Burnside coefficient system

    $\underline{A}(G/H) = A(H)$

The analogue

non-equivariant	equivariant
$\pi_*(X)$	$\underline{\pi}_*(X) : \mathrm{Orb}_G \to \mathrm{Ab}$
coefficient $A \in \mathrm{Ab}$	$\underline{A} : \mathrm{Orb}_G \to \mathrm{Ab}$.
$H^*(X;A)$, $H_*(X;A)$	$H^*(X;\underline{A})$, $H_*(X;\underline{A})$

Bredon cohomology

- non-equivariant cohomology (cellular)
  - CW complex X
  - Chain complex $C_n = C_n(X_n, X_{n-1})$
  - $\mathrm{Hom}(C_n, A)$ $\quad A \otimes C_n$ for homology
  - Take $H_\bullet$
- Bredon cohomology
  - G-CW complex X $\quad$ note: G-CW str. is compatible with G-action
  - $\underline{C}_n(X)(G/H) = H_n(X_n^H, X_{n-1}^H; \mathbb{Z})$ $\quad$ contravariant func.
  - $\mathrm{Hom}_{\text{coeff sys}}(\underline{C}_n, \underline{A}) \in Ab$ $\quad \underline{A} \otimes \underline{C}_n$ for homology, covariant
  - Take $H_\bullet \rightsquigarrow H^*(X; \underline{A})$

Axiomatic cohomology theory

- Definition:

  A generalized reduced Bredon cohomology theory is

  - a functor: $\widehat{H}_G^* : Ho(GTop_*) \longrightarrow Ab^{\mathbb{Z}}$
  - with isomorphisms: $\sigma^n : \widetilde{H}_G^n \xrightarrow{\cong} \widehat{H}_G^{n+1} \circ \Sigma$

  such that the following axioms are satisfied:

  - (additivity) sends $\vee$. products to $\prod$
  - (exactness) sends cof seq to exact seq
  - (dimension) $\widetilde{H}_G^*(G/H_+) \cong \begin{cases} 0 & * \neq 0 \\ A(G/H) & * = 0 \end{cases}$ A is a coeff system.

Properties

- Theorem: for a fixed coefficient system, there exists a unique generalized reduced Bredon cohomology functor.
- Bredon cohomology for constant coefficient system:

$$H^*_G(X; \underline{A}) \cong H^*(X/G; A)$$

By checking axioms + uniqueness.

An exercise: Smith theory

- Around 1940, P.A. Smith proved a number of theorems connecting $H^*(X)$ and $H^*(X^G)$ for a G-complex X.
- "...(our) proof is an almost trivial exercise in the use of Bredon cohomology" in the Alaska lecture notes.

Smith theory: statement

- G: a finite p-group for a prime p.
- X: a finite dimensional G-space, with finite dimensional mod p homology groups.

Theorem (Smith) The following hold.

$$\tilde{H}_*(X;\mathbb{F}_p) = \begin{cases} 0 & * \neq n \\ \mathbb{F}_p & * = n \end{cases}$$

- $\chi(X) \equiv \chi(X^G)$ mod p.
- If X is a mod p homology ical n-sphere, then X^G is either empty or a mod p homology ical m sphere for some $m \leq n$.

Proof

Assume $G = \mathbb{Z}/p$.

- Consider $X^G \to X \to FX := X/X^G$.

  free outside the base point

- By checking the axioms:
  - $\tilde{H}^q(FX/G)$ $\quad H^q(X)$ $\quad H^q(X^G)$

    By checking axioms, all are Bredon cohomology

    we take the corresponding coefficient systems

  - L $\quad M$ $\quad N$

Compute the coefficient system

coefficient systems.	space	G/G	G/e
L	$\tilde{H}^*(FX/G)$	0	$\mathbb{F}_p$
M	$H^*(X)$	$\mathbb{F}_p$	$\mathbb{F}_p[G]$
N	$H^*(X^G)$	$\mathbb{F}_p$	0
I		0	aug ideal of $\mathbb{F}_p[G]$
I^n		0	$(\quad)^n$

Fact $I^p = 0$, $I^{p-1} = L$

Short exact sequences

coefficient systems.		G/G	G/e
L	$\tilde{H}^*(FX/G)$	0	$\mathbb{F}_p$
M	$H^*(X)$	$\mathbb{F}_p$	$\mathbb{F}_p[G]$
N	$H^*(X^G)$	$\mathbb{F}_p$	0
I		0	aug ideal of $\mathbb{F}_p[G]$
I^n		0	$(\quad)^n$

Relations:

- $0 \to \underset{FX/G}{L} \to \underset{X}{M} \to \underset{X^G}{N} \oplus I \to 0$

$$\cdots \to H^n(X) \to H^n(X^G) \oplus H^n_G(X;I) \to \tilde{H}^{n+1}(FX/G) \to \cdots$$

$$\underbrace{\dim H^n(X^G)}_{\dim_n N} + \underbrace{\dim H^n_G(X;I)}_{\dim_n I} \le \dim \tilde{H}^{n+1}(FX/G) + \dim H^n(X)$$

① $\dim_n N + \dim_n I \le \dim_{n+1} L + \dim_n M$

- $0 \to I \to \underbrace{M \to N \oplus L}_{\text{connecting}} \to 0$

② $\dim_n N + \dim_n L \le \dim_n M + \dim_{n+1} I$

- $0 \to I^{n+1} \to I^n \to L \to 0, \quad 1 \le n \le p-1$

③ $\chi(I^n) = \chi(I^{n+1}) + \chi(L)$

- $\Sigma ③ + \chi(①)$

$$\cancel{\sum_{1\le n\le p-1} \chi(I^n)} + \chi(M) = \cancel{\sum_{1\le n\le p-1} \chi(I^{n+1})} + \chi(L)\cdot(p-1) + \chi(L) + \chi(N) + \cancel{\chi(I)}$$

$$\Rightarrow \underbrace{\chi(M)}_{\chi(X)} = \underbrace{\chi(N)}_{\chi(X^G)} + p\,\underline{\chi(L)}$$

$\sum_n (① + ②)$ $\quad \sum_n \dim_n \underset{X^G}{N} \le \sum_n \dim_n \underset{X}{M}$

$= 2$

$\sum_n \dim_n X^G \le 2$

Back to the proof

We want:

- $\chi(X) \equiv \chi(X^G)$ mod p.
- If X is a mod p homology n-sphere, then X^G is either empty or a mod p homology m sphere for some $m \leq n$.

Thank you!

Talk 14
Duality

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Duality: the general definition

- $(\mathscr{C}, \wedge, \mathrm{S})$: a symmetric monoidal category
- Definition: dual pairs

  (X, DX) is a dual pair if $\exists$

  - evaluation map $\mu : X \wedge DX \to S$
  - coevaluation map $\eta : S \to DX \wedge X$
  - diagrams commute :

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}\wedge\eta"] \arrow[rr, bend right, equal] & X \wedge DX \wedge X \arrow[r, "\mu\wedge\mathrm{id}"] & X \end{tikzcd}$$

+ the symmetric ones

Duality in the stable category

- $X \subset \mathbb{R}^n \to S^n$: a proper subspace.
- X^c: the compl~~om~~ement. lem
- Theorem (Spanier–Whitehead duality)
  Let $X \subset S^n$ be a proper compact subspace. Then $\Sigma^{1-n}X^c$ is the dual of X_+.

  Unstable : X_+ , X^c are $(n-1)$-dual : $X_+ \wedge X^c \to S^{n-1}$
  + coevaluati...

  pass to stable : drop the choice of n.
- Some results:
  - CW cplx are dualizable
  - $[A, B \wedge DC] \cong [A \wedge C, B]$ iso with some some finiteness condition
  - X finite spec $\rightsquigarrow E_*(X) \cong E^{-*}(DX)$

Specializing to manifolds

- M: a compact manifold without boundary
- TM: the tangent bundle
- Theorem (Atiyah Duality)
  We have $D(M_+) \simeq Th(-\mathrm{T}M)$.
- have a version of AD when M has boundary)
- Sketch proof: Use SW-duality.

1) $M \hookrightarrow \mathbb{R}^n$

2) $\Sigma^{1-n} M^c = \Sigma^{-n}(\mathbb{R}^n / \mathbb{R}^n - M) = \Sigma^{-n}(N / N - M)$

$\mathbb{R}^n - M \to \mathbb{R}^n \to \Sigma \mathbb{R}^n - M$

tubular nbhd.

$= \Sigma^{-n} N / \partial N = \Sigma^{-n} Th\nu$

normal bundle

$\nu \oplus TM = \mathbb{R}^n \rightsquigarrow D(M_+) \simeq Th(-TM)$

From Atiyah duality to Poincaré duality

- E: a cohomology theory (or the representing spectrum)
- M: compact manifold, dimension n, without boundary
- Theorem (Poincaré duality)
  M is E-orientable. Then $E^r(M_+) \cong E_{n-r}(M_+)$ for any r.
- E-orientation: a class $\mu \in \widetilde{E}^n(Th(TM))$ s.t.
  $\forall\, x : * \to M$, $x^*\mu \in \widetilde{E}^n(S^n) \cong \widetilde{E}_0(S^0) \cong \pi_0(E)$
  is the unit
- Sketch proof: use Thom isomorphism.
  $$E_*(M_+) = E^{-*}(DM_+) \overset{AD}{=} E^{-*}(Th(-TM))$$
  $$\underset{\text{Thom iso}}{=} E^{-*-(-n)}(M_+) = E^{n-*}(M_+)$$

Equivariant story

need orientation class

- Non-equivairantly, Poincaré Duality = Atiyah Duality + Thom iso
- Equivariant Atiyah Duality
- Equivariant Thom iso / orientation class
- The issues:

  - $(B)^H$ is not nec. connected.
  - $E \to B$ a G-fibration, isotropy grp of $x \in B$ is $H \rightsquigarrow F_x$ is H-space

The G-connected case

- E_G^*: an equivariant cohomology theory (or the spectrum)
- $p : E \to B$ a n-spherical G-bundle, B is G-connected

  B^H is connected. for every H
- Definition: E_G^* orientation class
  - Choose $x \in B^G$. Let V be the G-rep s.t $S^V = F_x$.
  - An E_G^* orientation class is $\mu \in \tilde{E}_G^V(Th\,p)$, s.t.

$\forall\, y \in B^H$ viewed as $y : G/H \to B$, the pullback $y^*\mu \in \tilde{E}_G^V(G/H_+ \wedge S^V) \cong \tilde{E}_G^0(G/H_+) \cong \pi_0^H(E)$ is a unit

- This gives Thom iso

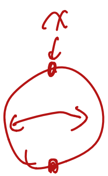
$$E_G^W(B) \longrightarrow E_G^{W+V}(Th\,p).$$

The general case

- Idea: make a local G-connected space for each point

  the connected component of B^K containing x

- Construction:
  - Fix $x \in B^H$
  - A presheaf $\Phi_x(H/K) = (B^K, x)$ for $\forall K \leq H$
  - (Elmendorf thm) $\rightsquigarrow$ a H-space $\bar{B}(x)$



$(\bar{B}(x))^{C_2} \simeq *$

$(\bar{B}(x))^{e} \simeq S^1$

- This assembles into a space over $\Pi_G(B)$

  Equivariant fundamental groupoid of B

- $\exists$ a natural map $\bar{B} \to B$
  induced by $(B^K, x) \hookrightarrow B^K$.

Orientation classes: the general definition

- Idea: a family indexed over the fundamental groupoid of the base space
- Definition:
  - A family of classes
  - $\forall x \in B^H$, a $\mu(x) \in \tilde{E}_a^{V(x)}(Th(p(x))$, s.t. — fiber at x — $\bar{E}(x) \to \bar{B}(x)$
  - $\mu(x)$ satisfies the pullback condition
- Thom isomorphism: A family of isomorphism.

The geometric definition

- There is a definition of orientation in a more geometric fashion.
- Non equivariant: the induced automorphism of fiber is independent of the choice of path.
- $\mathcal{F}$ the cat of fibers.

  A fiber bundle $p: E \to B$ defines a functor

  $p^*: \Pi(B) \longrightarrow h\mathcal{F}$
- This definition can be made equivariantly, with fundamental groupoid replaced by equivariant fundamental groupoid.
- Equivariant Geometric orientability = homological orientability for the Burnside coefficient system.

  reference: (Costenoble-May-Waner)

Thank you!