G finite group.

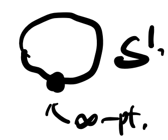
V G - representation. V vector space with $G \times V \to V$ linear

$\Leftrightarrow$ group homomorphism $G \to GL(V)$

Def. representation spheres.

$S^V :=$ one point compactification.

$$\mathrm{Thom}\left(\begin{array}{c} V \\ \downarrow \\ pt \end{array} \right) = S^V$$



e.g. G acts trivially on $\mathbb{R}^n$

($G \times \mathbb{R}^n \to \mathbb{R}^n$ $\Leftrightarrow$ $G \to GL_n(\mathbb{R})$)

$(g, \vec{x}) \to \vec{x}$ $\qquad$ $g \mapsto e$

$S^{\mathbb{R}^n} = S^n \circlearrowleft G$ acts trivially.

Grading over $\{V \mid V \text{ } G\text{-repn}\}$ $\qquad$ $RO(G)$-grading.

equivariant homotopy grps. $\pi_*^H X = [G/H_+ \wedge S^n, X]_G$

$$\pi_V^H X = [G/H_+ \wedge S^V, X]_G$$

V is an H-repn

- w.e. $f: X \to Y$ is a w.e. if $\pi_* f_H : \pi_* X^H \to \pi_* Y^H$ is an iso.
- G - CW complex.

  cells. $G/H_+ \wedge S^n$

⇝ compute Bredon cohomology theories

⇒ solve Smith theorems.

Motivation: ① G - manifold. $G/H_+ \wedge S^V$

Orientation / duality. needs to index over V.

② cohomology theories ⇝ Spectra.

"stable space"

equivariant cohomology theories → represented by G - Spectra.

G - space.

$H^*(X, \underline{A})$ extends to $RO(G)$ - grading.

↑ coefficient system ($H^V(X, \underline{A})$?)

$\underline{A}: O_G \to Ab$.

iff. $\underline{A}$ extends to a Mackey functor.

G - Spectra. $H^*(X, \underline{A})$ extends to a Mackey functor.

e.g. S^V $G = C_2$, $\sigma = \mathbb{R} \circlearrowleft C_2$

$x \to -x$

S^σ

$H^*(S^\sigma, \underline{A})$

$\cup$

y

$\underline{A} = \mathbb{Z}$ G - module M.

$\underline{M}$ coefficient system.

contravariant functor $\mathcal{O}_G \to Ab$.

$G/H \mapsto M^H$.

$H \subset K$ $\quad G/H \xrightarrow{proj} G/K \rightsquigarrow M^K \to M^H$

e.g. take $M = \mathbb{Z}$ w/ G trivial action

$\underline{\mathbb{Z}}$

$G/e \times S^1 \quad G/G \times S^0$

$X^1, X^0 \quad X^0$

$$\begin{array}{ccc} C_2/C_2 & 0 & \to \mathbb{Z} \oplus (\mathbb{Z}) \\ \downarrow & \downarrow & \downarrow \\ C_2/e & \mathbb{Z} \oplus \mathbb{Z} & \to \mathbb{Z} \oplus (\mathbb{Z}) \end{array}$$

$H^1((S^1)^{C_2}) = 0$

$H^1(S^1) = \mathbb{Z}$

Ex. S^n antipodal action.

$H_*(S^n_{\circlearrowleft}, \underline{\mathbb{Z}}) = ?$

C_2 anti

$(x_1, \dots, x_{n+1}) \to (-x_1, \dots, -x_{n+1})$

Hint : $\mathbb{RP}^n$.

Transfer map (only exists in stable case)

Pontryagin-Thom construction.

Non equivariant : $M \hookrightarrow \mathbb{R}^n$

normal bundle. ν.

$$\begin{array}{ccc} Thom(\nu) & \leftarrow & S^{\mathbb{R}^n} = S^n \\ x & \leftarrow & x \in T \\ \infty & \leftarrow & x \notin T \end{array}$$

$Thom\left(\begin{array}{c}\mathbb{R}^n \times M \\ \downarrow \\ M\end{array}\right) = S^n \wedge M_+$

tr : $S^n \to Thom(\nu) \to Thom(\nu \oplus t_M)$

$\nu \hookrightarrow t_M \oplus \nu$

tangent bundle of M

Ex. γ trivial n-dim bundle on M,

then. $Thom(\gamma) = M_+ \wedge S^n$.

tr : $S^n \longrightarrow S^n \wedge M_+ \to S^n$

equivariant case.

$$M \hookrightarrow V$$

G-space G-rep.

analogue to the above

$$S^V \to S^V \wedge M_+ \dashrightarrow S^V.$$

Take. $M = G/H$

$$S^V \to S^V \wedge G/H_+$$

$\Downarrow$ If desuspension V is allowed.

$$S^0 \to G/H_+$$

← there is NO G-equivariant map

but there is a G-eq. $S^V \wedge S^0 \to S^V \wedge G/H_+$!

$H \subset K$.

Stable tr: $G/K_+ \to G/H_+$

$\leftarrow$ proj

Mackey functor: $B_G \to Ab$.

↗.

coefficient system. / $Ob_G \to Ab$

obj finite G-set.

G/H

morph $Hom_{B_G}(G/H, G/K)$

$= [G/H, G/K]_G$.

stable G-groupoid. →

in particular, there are transfer maps.

$Hom_{Ob_G}(G/H, G/K)$

$= \{G/H, G/K\}$

G-group

$G/K \to G/H$

$H \subset K$.

Def. ? $B_G := Span^+(GFin)$,

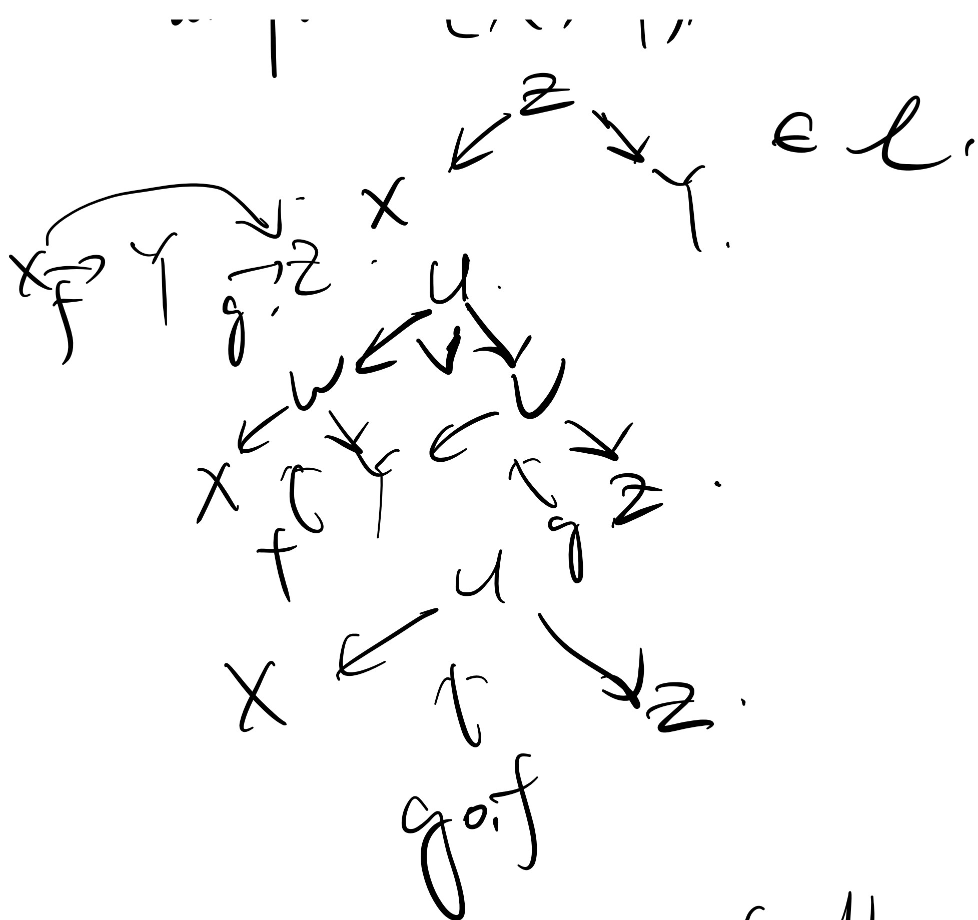
we will see

Finite G-set

Def : $Span(\mathcal{C})$, $\mathcal{C}$. small. cat. finite product coproduct.

obj = obj($\mathcal{C}$)

morphism (X, Y).



input $\mathcal{C}$. cat { small, finite product / coproduct.

operation Span ↓

output $\mathrm{Span}(\mathcal{C})$ cat.

in our case.

$\mathrm{Span}(G\mathrm{Fin})$

$G/H \xrightarrow{\text{proj}} G/K$ in $G\mathrm{Fin}$

$\ldots H_+ \quad \cdot \quad K_+ \quad \cdot \quad \ldots$

$H \subset K$

In $\mathrm{Span}(G\widehat{\mathrm{Fin}})$

$$G/H_+ \xlongequal{\quad} G/H_+ \longrightarrow G/K_+$$

$$G/H_+ \longrightarrow G/K \xlongequal{\quad} G/K_+$$

Exercise:

write out the

$$F : \mathrm{Span}^+(G\mathrm{Fin}) \to \mathrm{Ab}.$$

extend $\mathrm{Coeff} \longrightarrow \mathrm{Ab}$.

for the case: $\underline{\mathbb{Z}}$

$$X \xleftarrow{} A \xrightarrow{f} Y, \qquad X \xleftarrow{} B \xrightarrow{g} Y$$

$$f+g : X \leftarrow A \sqcup B \rightarrow Y.$$

Thm. In G-space, $H^*_G(-;\mathcal{A})$ extends to an $RO(G)$-graded theory iff the coefficient system $\mathcal{A}$ extends to a Mackey functor.

Rk. • $\underline{\mathbb{Z}}$ constant coefficient system.
$\rightsquigarrow$ Mackey functor.

$\Rightarrow$ $H^*(-,\underline{\mathbb{Z}}) \rightsquigarrow H^\star(-,\underline{\mathbb{Z}})$

• G-spectra. H^*_G extends to an $RO(G)$ graded.

e.x. Constant <u>mackey functor</u>. $\underline{\mathbb{Z}}$

$H \subset K$.

$$\underline{\mathbb{Z}}(G/H) = ? \qquad (\mathbb{Z})^H = \mathbb{Z}$$

res ↓ ↑ trans. $\qquad \times|K/H|$ ↓ ↑ id.

$$\underline{\mathbb{Z}}(G/K) \qquad (\mathbb{Z})^K = \mathbb{Z}$$

Ex. $H^\star(S^V,\underline{\mathbb{Z}})$
$H^*(S^V,\underline{\mathbb{Z}})$ computation.

• $G = C_2 \quad V = n\rho_{C_2}$
↑ regular representation.

$H^\star(S^{n\rho_{C_2}},\underline{\mathbb{Z}})$

e.g. Burnside Mackey functor. A_G

$$A_G(G/H) = (\text{finite } H \text{ set}, \sqcup)^{\text{grp completion}}$$

$H \subset K$

$\downarrow \uparrow$ G/K

- K-set X can be regarded as H-set.
- H-set $X \to K \times_H X$

e.g.

$$R(G)(G/H) = (\text{finite } H\text{-rep}, \oplus)^{\text{grp completion}}$$

- $V \to i_H V$
- $W \xrightarrow{\text{induction}} \mathbb{Z}[K] \otimes_{\mathbb{Z}[H]} W$

W: H-rep. $\quad$ K-rep.

e.g. $\pi_n^H(X) := [G/H_+ \wedge S^n, X]_G$.

$\square$ ← vary.

$\pi_n(X)$

$$\underline{\pi}_n(X)(G/H) = \pi_n^H(X)$$

(Wirthmüller isomorphism). ← "orbits G/H self dual."

Advertise.

Application next week: Conner conjecture. ← transfer functor.

HHR. Mandey ...

Sp^G. G-spectra.

Recall

$\{X_i\}$ $\Sigma X_i \to X_{i+1}$.

realized $\wedge$:
- orthogonal.
- EKMM. (S-module)
- Symetric.

Non equivariant:

Orthogonal Spectra: Sp^O

$E: \mathcal{O} \to Top_*$

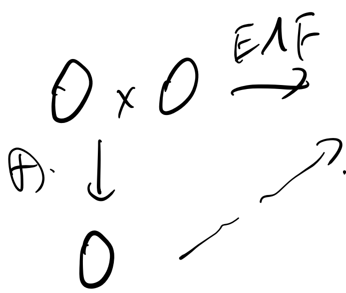
($\mathcal{O}$: Top cat)

obj: finite dim inner product. vector spaces

morphism: $\mathcal{O}(V,W) := Th(\xi)$ $\quad \xi = \{(\varphi, w \in W) \mid w \in \varphi(V)^\perp\} \downarrow L(V,W)$

$L(V,W) = \{$linear isometric embedding $V \hookrightarrow W\}$

Def $\wedge$: $E \wedge F$



Day convolution.

Left Kan extension

unpack the definition.

[obj.]

$$\{ X_n , \ \Sigma X_n \to X_{n+1} \}.$$
$$O(n) \times X_n \to X_n.$$

eg. suspension map: $X_n \wedge S^m \to X_{n+m}$, where $O(n) \times O(m)$ acts on each factor and $O(n+m)$ acts on the target, with $O(n) \times O(m) \hookrightarrow O(n+m)$ as $\begin{pmatrix} O(n) & 0 \\ 0 & O(m) \end{pmatrix}$

is $O(n) \times O(m)$ - equivariant map.

[Morph.] $f : X \to Y$ $\sigma_n : X_* \wedge S^n \to X_{*+n}$.

$f_n : X_n \to Y_n$, $f_{*+n} \sigma_n = \sigma_n f_*$

↓ Analogue. G - spect.

Def. universe U. G - repn.

st. if $V \subset U$ subrepn.

then. U contains infinite copies of V.

U is complete if U contains all irreducible G - repn.

eg. $U = \{ \mathbb{R}^{\infty} \}$ trivial repn.

$\mathcal{U} = \rho_G^{\oplus\infty}$ complete.

↖ regular representation

$(G = C_2 \quad 1, \sigma \quad \rho_{C_2} = 1+\sigma)$

G-Spectra:

obj: $\{X_n\}$ $O(n) \times G$-action.

$\sigma_n : X_n \wedge S^1 \to X_{n+1}$

+ G-equivariant.

$O(n) \times O(1)$-equivariant!

$(\Sigma^V : X \to X \wedge S^V)$

$\Omega^V : X \to F(S^V, X)$

Ex. Adjunction.

Above information is enough to give all $RO(G)$-grading.

$$X_V := L(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$$

↑ linear isometric $f: \mathbb{R}^n \to V$.

$$gf : \mathbb{R}^n \xrightarrow{g \in O(n)} \mathbb{R}^n \xrightarrow{f} V.$$

e.g. $\mathcal{S} = \{ S^0, S^1, S^2, \ldots \}$ (homogeneous), with $O(2)$ acting on S^2

$S^n = S^{\mathbb{R}^n} \supset O(n)$, with $O(n)$ acting on S^n.

e.g. $\mathcal{S}^G = \{ S^0, S^1, \ldots \}$

$O(n) \times G \curvearrowright S^n$.

$O(n)$: as above; G: trivial.

! warning. $\mathcal{S}$ has nontrivial action.

$$\mathcal{S}(V) = L(\mathbb{R}^n, V)_+ \wedge_{O(n)} S^n.$$

$\curvearrowright$

G acts nontrivial.

(if G acts on V nontrivial.)

G_1 : $\mathbb{R}$ $2G_2$.

$S^\sigma \rightsquigarrow$ 

$G/G_+ \wedge S^0$	0
$G/e_+ \wedge S^1$	1