Equivariant spectra

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Plan
1. A flash introduction to Lewis-May G-spectra
2. Wirthmüller isomorphism: induced and coinduced
3. Geometric fixed points and the isotropy separation
4. Homotopy fixed points and Tate diagram

1. A FLASH INTRODUCTION TO LEWIS-MAY G-SPECTRA

G : compact Lie group

- The (enlarged) equivariant Spanier-Whitehead category

SW^G : obj. (X, V) "$\Sigma^{-V} X$"

X: based finite G-CW complex; V: G-rep (finite)

$$\text{mor}\, SW^G\big((X,V),(Y,W)\big) = \operatorname{colim}_{V'} \left[\Sigma^{V'+W} X, \Sigma^{V'+V} Y\right]_G$$

Finite G-CW complex has dual objects in SW^G.

SW^G does not have all colimit and limit.

$\Rightarrow$ need a better model.

- Universe : $\mathcal{U}$: a infinite dim'l G-rep.
  - $\mathbb{R} \subseteq \mathcal{U}$
  - $V \subseteq \mathcal{U} \Rightarrow \mathcal{U}$ has infinite many copies of V.

G-prespectra indexed on $\mathcal{U}$: $V \subseteq \mathcal{U} \rightsquigarrow$ based G-space $E(V)$

E $\qquad V \subseteq W \subseteq \mathcal{U} \rightsquigarrow \sigma_{V,W} : S^{W-V} \wedge E(V) \to E(W)$
+ compatibility

$\Rightarrow GP\mathcal{U}$ $\qquad f: E \to F \rightsquigarrow f(V) : E(V) \to F(V)$ compatible with $\sigma_{V,W}$.

G-spectra indexed on $\mathcal{U}$: $\tilde{\sigma}_{V,W} : E(V) \to \Omega^{W-V} E(W)$ homeomorphism for all V, W.

$\Rightarrow GS\mathcal{U}$

$$GP\mathcal{U} \underset{\ell}{\overset{L}{\rightleftarrows}} GS\mathcal{U}$$

L: spectrification

$$GP\mathcal{U} \underset{L}{\overset{\;}{\rightleftarrows}} GS\mathcal{U}$$

$\hookleftarrow$ inclusion

Change of universe $\quad f: \mathcal{U} \to \mathcal{U}'$ (G-isometric embedding)

$$f_* : GS\mathcal{U} \rightleftarrows GS\mathcal{U}' : f^*$$

$$(f^*E)(V) := E(f(V)) \qquad V \leq \mathcal{U}$$

$$(f_*E)(V') := E(V) \wedge S^{V'-f(V)} \qquad V' \leq \mathcal{U}'$$

$$V = f^{-1}(V' \cap f(\mathcal{U}))$$

and then spectrify.

useful in two scenarios:

(1) $\mathcal{U} \cong \mathcal{U}'$ $\qquad \mathcal{L}(\mathcal{U}, \mathcal{U}')^G \simeq *$

f_* and f'_* give equivalent functors.

(2) $i: \mathcal{U}^G \to \mathcal{U}$. $\mathcal{U}$ is a complete G-universe (contains all irreps)

G-spectra indexed on $\mathcal{U}^G \leftrightarrow$ naive G-spectra

G-spectra indexed on $\mathcal{U} \leftrightarrow$ genuine G-spectra

$$i_* : \underset{\text{naive}}{GS\mathcal{U}^G} \rightleftarrows \underset{\text{genuine}}{GS\mathcal{U}} : i^*$$

- smash product:

$E, F \in GS\mathcal{U} \Rightarrow E \bar{\wedge} F \in GP(\mathcal{U} \oplus \mathcal{U})$

$\Rightarrow$ spectrification, $L(E \bar{\wedge} F) \in GS(\mathcal{U} \oplus \mathcal{U})$

use $f: \mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$

$\Rightarrow f_* L(E \bar{\wedge} F) \in GS\mathcal{U}$

$\wedge$ is only unital, associative and commutative in the homotopy category.

- Σ^∞_G

- function spectra $\quad E, F \in GS\mathcal{U} \Rightarrow F(E, F) \in GS\mathcal{U}$.

for $X \in GTop$, $F(X, F)(V) = Map(X, F(V))$

for $E \in GS\mathcal{U}$, need to use $f: \mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$.

for $E \in GSU$, need to use $f: U \oplus U \to U$.

- $GSU(f_*(E \wedge E'), E'') \cong GSU(E, F(E', f^*E''))$

change of universe

- We have the notion of G-CW spectra & G-CW approximation.

$[E, F]_G :=$ G-homotopy classes of maps $\Gamma E \to \Gamma F$

G-CW approximation.

n can be negative

Definition 1.3. We define the H-equivariant homotopy groups of a G-spectrum E to be

$$\pi_n^H(E) = [G/H_+ \wedge S^n, E]_G.$$

This assembles into a coefficient system

$H \subseteq K \Rightarrow G/H \to G/K$

$$\underline{\pi}_n(E) : \mathscr{O}_G^{op} \to \mathrm{Ab}, \underline{\pi}_n(E)(G/H) = \pi_n^H(E).$$

Theorem $f: E \to E'$. $f(V)$ is a weak equivalence for all V

$\Leftrightarrow$ $\pi_n^H(f)$ is an isomorphism for all n, $H \leq G$.

Fixed point spectra $D \in GSU^G$. $D^G \in SU^G$

naive spectra

$D^G(V) = (D(V))^G$ $V \subseteq U^G \subseteq U$

$\Rightarrow D^G$ is automatically a spectrum.

$E \in GSU$ $E^G := (i^*E)^G \in SU^G$

genuine spectra

Remark

$GSU^G(C, D) \cong SU^G(C, D^G)$

$GSU(i_*C, E) \cong SU^G(C, E^G)$

In Lewis-May G-spectra, $(-)^G$ is homotopical.

In other models, $(-)^G$ categorical fixed point is not homotopical, so one also has F^G derived fixed point.

What about orbit spectra?

$D \in GSU^G \Rightarrow D/G \in SU^G$ ① $D/G(V) = D(V)/G$

② spectrify

$E \in GSU \Rightarrow$ The guess $L(i^*E)/G$ is not useful.

$E \in GS\mathcal{U} \Rightarrow$ The guess $L(i^*E)/G$ is not useful.

Instead, if E is G-free, then there is a G-free naive spectrum D such that $i_* D \simeq E$.

We define $E/G := D/G$

Remark. $i^*E \not\simeq D$
$\quad \| $
$i^* i_* D$

For example, take $D = \Sigma^\infty G_+$

$i_* D = \Sigma^\infty_G G_+$ suspension to the complete universe

$(i^* i_* D)^G = (\Sigma^\infty_G G_+)^G$

$\simeq \Sigma^\infty EG_+ \wedge_G G_+ \simeq \mathbb{S} \not\simeq *$

$W\{e\} = G/e = G$

So $i^* i_* D$ is not G-free.

So not $\simeq D$.

Slogan: Only take orbits of a G-free genuine spectra.

Relation to suspension functor

Tom-Dieck splitting

$$(\Sigma^\infty_G A)^G \simeq \bigvee_{(H)\subseteq G} \Sigma^\infty EWH_+ \wedge_{WH} A^H$$

- (Co)homology groups

$X, E \in GS\mathcal{U}$

$E^G_n(X) := \pi_n(E \wedge X)^G$

$E^n_G(X) := \pi_n F(X, E)^G$

- Transfer map, geometric construction (lecture 16)

$H \subseteq K \Rightarrow \underline{\pi}^K_n(E) \to \underline{\pi}^H_n(E)$ restriction map

$\Rightarrow$ wrong way map in stable category

$\Rightarrow \underline{\pi}^H_n(E) \to \underline{\pi}^K_n(E)$ transfer map

$\Rightarrow \underline{\pi}_n(E)$ is a Mackey functor for a genuine G-spectrum E.

2. Wirthmüller isomorphism: induced and coinduced

$\mathcal{U}$: complete G-universe $\quad H \subseteq G \rightsquigarrow$ complete H-universe, still denoted as $\mathcal{U}$.

$\Rightarrow \quad \mathrm{Res}^G_H : GS\mathcal{U} \longrightarrow HS\mathcal{U}$

$\Rightarrow$ left adjoint $\quad G_+ \wedge_H - \quad$ induced G-spectra

right adjoint $\quad F_H(G_+, -) \quad$ coinduced G-spectra

$L(H)$: the tangent of eH in G/H. When G is discrete, $L(H) = \{0\}$

Theorem 2.1 (Wirthmüller Isomorphism)**.** *[GHT, Theorem 3.2.15]. Let H be a subgroup of G and X an H-spectrum. Then there is a natural weak equivalence of G-spectra*

$$F_H(G_+, \Sigma^{L(H)} X) \to G_+ \wedge_H X.$$

Consequence. 1. $X = \mathbb{S}_H \quad F_H(G_+, \mathbb{S}_H) \simeq G_+ \wedge_H \mathbb{S}_H$

G finite

$$\begin{array}{ccc} F_H(G_+, \mathbb{S}_H) & \simeq & G_+ \wedge_H \mathbb{S}_H \\ \wr\mathrm{S} & & \wr\mathrm{S} \\ F(G/H_+, \mathbb{S}_G) & & G/H_+ \wedge \mathbb{S}_G \simeq G/H_+ \\ \wr\mathrm{S} & & \\ D(G/H_+) & & \end{array}$$

$D(G/H_+)$ Spanier–Whitehead dual

$\Rightarrow G/H_+$ are self-dual

2. Take $X = X \wedge E$. $\quad E \in GS\mathcal{U}$

$$\begin{array}{ccc} \left(F_H(G_+, \Sigma^{L(H)}(X \wedge E))\right)^G & \simeq & \left(G_+ \wedge_H (X \wedge E)\right)^G \\ \wr\mathrm{S} & & \wr\mathrm{S} \\ \left(\Sigma^{L(H)} X \wedge E\right)^H & & \left((G_+ \wedge_H X) \wedge E\right)^G \end{array}$$

$\Rightarrow$ Take π^G_n,

$$\boxed{E^H_*(\Sigma^{L(H)} X) \cong E^G_*(G_+ \wedge_H X)}$$

cohomology

$$\boxed{E^*_H(X) \cong E^*_G(G_+ \wedge_H X)}$$

$$\begin{array}{ccc} E^*_H(X) & \cong & E^*_G(G_+ \wedge_H X) \\ \wr\mathrm{S} & & \| \\ [X, E]_H & \cong & [G_+ \wedge_H X, E]_G \end{array}$$

- Transfer map, categorical construction

Goal: from $G/K \xrightarrow{f} G/H$, want $G/H \to G/K$

$$[G/K, G/H]_G \cong [\mathbb{S}, D(G/K) \wedge G/H]_G \cong [D(G/H), D(G/K)]_G$$

$\Downarrow$ $\quad\quad$ $\wr\mathrm{S}$ self-dual

f - - - - - - - - - - - - - - - - - - $\to$ $\exists$ a map

$$[G/H, G/K]_G$$

$\Downarrow$

3. Geometric fixed points and the isotropy separation

motivation: $(\Sigma^\infty_G X)^G \not\simeq \Sigma^\infty X^G$

but we have $\Phi^G : GSU \to SU^G$

- $\Phi^G(\Sigma^\infty_G X) \simeq \Sigma^\infty(X^G)$
- $\Phi^G(X \wedge Y) \simeq \Phi^G(X) \wedge \Phi^G(Y)$

- The conceptual construction

a family of subgroups $\mathcal{F}$ is a collection of subgroups of G that is nonempty and closed under subgroups and conjugation.

$\mathcal{F}$ is supposed to govern the isotropy subgroups of a G-space/spectra

$\Rightarrow \exists$ a G-space $E\mathcal{F}$ such that $E\mathcal{F}^H \simeq \begin{cases} * & H \in \mathcal{F} \\ \phi & H \notin \mathcal{F} \end{cases}$

Proposition: $f: E \to F$ is a $\mathcal{F}$-equivalence

$\Rightarrow$ $E \wedge E\mathcal{F}_+ \to F \wedge E\mathcal{F}_+$ is a G-equivalence.

Example. $\mathcal{F} = \{e\}$. $E\mathcal{F} = EG$

$\Rightarrow$ $f: E \to F$ G-map, underlying w.e.

$\Rightarrow$ $E \wedge EG_+ \to F \wedge EG_+$ is a G-equivalence.

Example. $\mathcal{F} = \mathcal{P} = \{$all proper subgroups of $G\}$.

$E\mathcal{P}_+ \to S^0 \to \widetilde{E\mathcal{P}}$ ← isotropy separation sequence

Definition. $\Phi^G(E) = (\widetilde{E\mathcal{P}} \wedge E)^G$.

ρ_G: regular G-representation. $\bar{\rho}_G$: reduced

Take $\operatorname{colim}_n \underset{\substack{\uparrow \\ \text{unit sphere}}}{S}(n\bar{\rho}_G) \simeq E\mathcal{P} = S(\infty\bar{\rho}_G)$

$S(V)_+ \to S^0 \to S^V$

$\widetilde{E\mathcal{P}} = S^{\infty\bar{\rho}_G}$

verify: $(S^{\infty\bar{\rho}_G})^H \simeq S^\infty \simeq *$ for $H \subsetneq G$

$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}$

$(E\mathcal{F})^H \simeq \begin{cases} * & H \in \mathcal{F} \\ \phi & H \notin \mathcal{F} \end{cases}$ $\quad (\widetilde{E\mathcal{F}})^H \simeq \begin{cases} * & H \in \mathcal{F} \\ S^0 & H \notin \mathcal{F} \end{cases}$

- The point set construction

Set $\mathcal{U} = \infty \rho_G$. $X \in G\mathcal{P}\mathcal{U}$

prespectrum $\Phi^G(X)(V) = X(\rho_G \otimes V)^G$

$V \subseteq \mathcal{U}^G = \mathbb{R}^\infty$

fixed point $X(n)^G$

geometric f.p. $X(n\rho_G)^G$

then spectrify $\Rightarrow L\Phi^G \in S\mathcal{U}^G$.

Prop. The two definitions are equivalent. $\Phi^G L \simeq L\Phi^G$

Prop. $\Phi^G(\underline{\Sigma_G^\infty X}) \simeq \Sigma^\infty(X^G)$.

Pf. $\Phi^G(\Sigma_G^\infty X)(V) = (\Sigma_G X(V \otimes \rho_G))^G = (X \wedge S^{n\rho_G})^G = X^G \wedge S^n$

$V = n$ $\qquad = \Sigma^\infty X^G(n)$

$\Rightarrow L\Phi^G(-) \simeq \Sigma^\infty(X^G)$

Next goal: Φ^H.

Case 1. $N \triangleleft G$ normal.

$\mathcal{F}[N] = \{ K \leq G \mid N \nsubseteq K \}$ subgroups not containing N.

(NOT $\mathcal{F}(N) = \{ K \leq G \mid K \cap N = \{e\} \}$)

$\Rightarrow \Phi^N : GS\mathcal{U} \longrightarrow (G/N)S\mathcal{U}^N$

$E \longmapsto (E \wedge \widetilde{E\mathcal{F}[N]})^N$

Case 2. H general. $1 \to H \to N_G H \to W_G H \to 1$

$\Phi^H : GS\mathcal{U} \xrightarrow{\text{fgt}} (N_G H)S\mathcal{U} \xrightarrow{\Phi^H} (W_G H)S\mathcal{U}^H$

Theorem. $f: X \to X'$ of G-spectra is an equivalence if and only if $\Phi^H f : \Phi^H X \to \Phi^H X'$ is non-equivariantly an equivalence for all $H \leq G$.

4. HOMOTOPY FIXED POINTS AND TATE DIAGRAM

$E \in G\mathcal{SU}$

$\Rightarrow \quad E^{hG} = F(EG_+, E)^G$

$E_{hG} = EG_+ \wedge_G E$

isotropy separation

$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}} \qquad \mathcal{F} = \{e\} \quad E\mathcal{F} = EG$

$\Rightarrow (EG_+ \to S^0 \to \widetilde{EG}) \wedge X$

$\qquad (\qquad) \wedge F(EG_+, X)$

induced by $EG_+ \to S^0$

$$\begin{tikzcd}
EG_+ \wedge X \arrow[r] \arrow[d, "\simeq\ \mathrm{id}\wedge\varepsilon"] & X \arrow[r] \arrow[d, "\varepsilon"] & \widetilde{EG} \wedge X \arrow[d, "\mathrm{id}\wedge\varepsilon"] \\
EG_+ \wedge F(EG_+, X) \arrow[r] & F(EG_+, X) \arrow[r] & \widetilde{EG} \wedge F(EG_+, X)
\end{tikzcd}$$

① Left vertical $\simeq$: $\varepsilon: X \to F(EG_+, X)$ is an underlying equivalence

$\Rightarrow EG_+ \wedge \varepsilon$ is a G-equivalence $\quad \mathcal{F} = \{e\}$

② Adams isomorphism (G finite)

$(EG_+ \wedge X)^G \simeq EG_+ \wedge_G X \simeq X$

$$\begin{tikzcd}
X_{hG} \arrow[r] \arrow[d, "\simeq"] & X^G \arrow[r] \arrow[d] & (\widetilde{EG} \wedge X)^G \arrow[d] \\
X_{hG} \arrow[r, "\text{norm}"] & X^{hG} \arrow[r] & X^{tG}
\end{tikzcd}$$

$X^{tG} := (\widetilde{EG} \wedge F(EG_+, X))^G$ Tate fixed point.

Remark. When $G = C_p$ (cyclic group of prime order)

$\mathcal{P} = \{e\} \Rightarrow \widetilde{EG} = \widetilde{E\mathcal{P}}$

$$\begin{tikzcd}
X_{hC_p} \arrow[r] \arrow[d, "\sim"] & X^{C_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Phi^{C_p} X \arrow[d] \\
X_{hC_p} \arrow[r] & X^{hC_p} \arrow[r] & X^{tC_p}
\end{tikzcd}$$

Remark. V: G-module. $\Rightarrow H^n(G, V)$, $H_n(G, V)$

$N: H_0(G, V) \cong V_G \longrightarrow V^G \cong H^0(G, V)$

Notation: V: G-module $\Rightarrow$ $H^*(G,V)$, $H_*(G,V)$

$$N: H_0(G,V) \cong V_G \longrightarrow V^G \cong H^0(G,V)$$

$$[v] \longmapsto \sum_{g\in G} gv$$

$$\hat{H}^n(G,V) = \begin{cases} H^n(G,V) & n \geq 1 \\ \operatorname{coker} N & n=0 \\ \ker N & n=-1 \\ H_{-n-1}(G,V) & n \leq -2 \end{cases}$$

$$\pi_*(HM)^{tG} \cong \hat{H}^{-*}(G, M(G/e))$$

Mackey functor   G-module

Summary

$GS\mathcal{U}$	genuine	G-spectra	$\underline{\pi}_*$ Mackey functor
$GS\mathcal{U}^G$	naive	G-spectra	coefficient system

Fixed points

Geometric fixed points
Isotropy separation

Homotopy fixed points and homotopy orbits

Tate diagram

Wirthmüller iso   "induced = coinduced"

Adams iso   "orbits = fixed points"