Generators of $\pi_* BP^{((C_{2^n}))}$

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Compute $\pi_\star MU^{((G))}$ and $\pi_\star BP^{((G))}$ for $G = C_{2^n}$.

$C_2 \subseteq G$. $MU^{((G))} := N_{C_2}^{G} MU_{\mathbb{R}}$ $BP^{((G))} := N_{C_2}^{G} BP_{\mathbb{R}}$.

- choice of $MU_{\mathbb{R}}$ or $BP_{\mathbb{R}}$.

$MU_{\mathbb{R}}$: Real bordism ⇝ cplx bordism MU with cplx conjugate action.

☆ $\Phi^{C_2}(MU_{\mathbb{R}}) \simeq MO$ Φ^{C_2}: Geometric fixed points.

$\uparrow$

$Gal(\mathbb{C}/\mathbb{R}) \cong C_2$ $(\mathbb{C})^{C_2} \cong \mathbb{R}$.

The only geometric input in HHR.

$\Downarrow$

(Should) give All differentials in the slice spectral sequence.

$\boxed{\text{computation: e.g. Adams SS. Input: algebraic. Diff/Ext: geometric.}}$

Classically: $MU_{(p)} \simeq \vee \Sigma^{?} BP$ BP: Brown-Peterson spectrum at p.

$MO \simeq \vee \Sigma^{?} H\mathbb{F}_2$ $H\mathbb{F}_2$: E-M spectrum of $\mathbb{F}_2$

Put some rep. here.

C_2-equivariantly: $MU_{\mathbb{R}(2)} \simeq \vee \Sigma^{?} BP_{\mathbb{R}}$: Real Brown-Peterson spectrum at 2.

$$\begin{array}{ccc} \Phi^{C_2}(MU_{\mathbb{R}(2)}) & \simeq & \vee \Sigma^{?} \Phi^{C_2}(BP_{\mathbb{R}}) \\ \wr\!\wr & & \\ MO & \simeq & \vee \Sigma^{?} H\mathbb{F}_2 \end{array}$$

Thm (Araki)

- $\Phi^{C_2}(BP_{\mathbb{R}}) \simeq H\mathbb{F}_2$
- The splitting of $MU_{\mathbb{R}(2)}$ is compatible with the splitting of MO under Φ^{C_2}.

Working over $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$ is equivalent.

Problem: $MU_{\mathbb{R}}$ is a C_2-commutative ring spectrum.

$\Downarrow$

We can use norms. R ring

$C_2 \hookrightarrow \mathrm{Aut}(R)$ Transfer

$x \in R \quad x + \gamma x \in R^{C_2}$

$x \cdot \gamma x \in R^{C_2}$

norm.

$BP_{\mathbb{R}}$ is NOT a C_2-commutative ring spectrum.

[BP is not a commutative ring spectrum]

[Lawson]

$\Downarrow$

We might not be able to use norm.

We work with $BP_{\mathbb{R}}$ from now on.

Goal: $\pi_\star^{\mu} BP^{((G))}$ for $G = C_{2^n}$. [with G-action]

- Why?
- How?

$X \in Sp^G$. Questions to ask:

① What is X as a non-eq. spectrum?

② $G \curvearrowright \pi_* X$. what is this action?

$\Uparrow$

computes the homotopy fixed points SS - E_2

- Try to understand X^{hG}.

③ What is X as an H-spectrum for $H \subseteq G$?

⋮

Reason 1: It is the first thing we try to understand.

Reason 2: It is the foundation of the slice tower of $BP^{((G))}/MU^{((G))}$.

$\Updownarrow$

Postnikov tower.

Reason 3: · It will help in understanding the detection thm.

$\Uparrow$

How things are coming together.

Thm. For $G = C_{2^n}$ $\pi_*^{\mu} BP^{((G))} \cong \mathbb{Z}_{(2)}[G\cdot t_1^G, G\cdot t_2^G, \cdots]$

$|t_i^G| = 2(2^i - 1)$ $\qquad G = \langle \gamma \rangle.$

$$\left.\begin{array}{l} G\cdot x := \{x, \gamma x, \gamma^2 x, \cdots, \gamma^{2^{n-1}-1} x\}. \\ \qquad \gamma^{2^{n-1}} \cdot x = -x. \end{array}\right\} \text{a } G\text{-set.}$$

Example: • $G = C_2$.

$\pi_*^{\mu} BP_{\mathbb{R}} \cong \mathbb{Z}_{(2)}[t_1^{C_2}, t_2^{C_2}, \cdots]$

$\gamma t_i = -t_i$

$k_{\mathbb{R}}$: connective Real K-theory

Fact: $k_{\mathbb{R}} \simeq BP_{\mathbb{R}} / (t_2^{C_2}, t_3^{C_2}, \cdots)$

• $G = C_4$.

$\pi_*^{\mu} BP^{((C_4))} \cong \mathbb{Z}_{(2)}[C_4 \cdot t_1^{C_4}, C_4 \cdot t_2^{C_4}, \cdots]$

$\gamma(\gamma t_i) = -t_i$

$$\begin{tikzcd}
t_i \arrow[r, mapsto, "\gamma"] \arrow[dr, phantom, "C_4\text{-action}"] & \gamma t_i \arrow[d, "\gamma"] \\
-\gamma t_i \arrow[u, "\gamma"] & -t_i \arrow[l, mapsto, "\gamma"]
\end{tikzcd}$$

How to prove it?

Start with easy examples:

$G = C_2$.

History: · How to compute $\pi_* BP$? Steenrod algebra.

Milnor: compute $H^*(BP; \mathbb{F}_p)$ as A-module

then compute its Adams spectral sequence.

collapse at E_2.

Ref: On the cobordism ring Ω^* and a cplx analogue.

Thm (Milnor)

I. · $\pi_* BP$ is a polynomial ring with one generator in each degree of the form $2(p^i - 1)$

· $H_*(BP; \mathbb{Z}_{(p)}) := \pi_* H\mathbb{Z}_{(p)} \wedge BP$

$$\cong \mathbb{Z}_{(p)}[m_1, m_2, \cdots]$$

$$|m_i| = 2(p^i - 1)$$

· Given $\{v_i\} \subseteq \pi_* BP$ $|v_i| = 2(p^i - 1)$.

then $\{v_i\}$ is a set of poly. gen. of $\pi_* BP$

iff for each i

$$\begin{tikzcd}
\pi_{2(p^i-1)}BP \arrow[r] & H_{2(p^i-1)}(BP;\mathbb{Z}_{(p)}) \arrow[d] \\
& Q_{2(p^i-1)}H_*BP \cong \mathbb{Z}_{(p)}\langle m_i \rangle \arrow[d]
\end{tikzcd}$$

indecomposable quotient. $\rightsquigarrow$ $Q_{2(p^i-1)}H_*BP \cong \mathbb{Z}_{(p)}\langle m_i\rangle$

$$Q_{2(p^i-1)}H_*BP \to \mathbb{Z}/p, \qquad m_i \mapsto 1$$

image of v_i in $Q_{2(p^i-1)}$ generates

$$\mathrm{Ker}(Q_{2(p^i-1)} \to \mathbb{Z}/p) \cong \langle p m_i \rangle$$

II. Consider $\overset{\ell}{\wedge} BP$

- $H_*(\overset{\ell}{\wedge}BP;\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[m_{1,1}, m_{1,2}, \cdots, m_{1,\ell},$
  $m_{2,1}, m_{2,2}, \cdots, m_{2,\ell},$
  $\vdots$
  $]$.

$$|m_{ij}| = 2(p^i-1)$$

[Künneth formula: compute $H_*(X\wedge Y)$ from $H_*(X)$ and $H_*(Y)$].

$$\pi_*\overset{\ell}{\wedge}BP \hookrightarrow H_*(\overset{\ell}{\wedge}BP;\mathbb{Z}_{(p)})$$

- Given $\{v_{i,j}\}$ a set of elements in $\pi_* \overset{\ell}{\wedge} BP$.

  $i = 1, 2, 3, \cdots \qquad 1 \le j \le \ell$.

  $\{v_{ij}\}$ is a set of poly. gen. of $\pi_* \overset{\ell}{\wedge} BP$

  iff for each fixed i.

$$\begin{tikzcd} \pi_{2(p^i-1)} \overset{\ell}{\wedge} BP \arrow[r, hook] & H_*(\overset{\ell}{\wedge} BP; \mathbb{Z}_{(p)}) \arrow[r] & Q_{2(p^i-1)} \arrow[d] \\ & & \mathbb{Z}/p \end{tikzcd} \qquad m_{ij} \mapsto 1$$

$\{v_{ij}\}$ generates $\mathrm{Ker}(Q_{2(p^i-1)} \twoheadrightarrow \mathbb{Z}/p)$

Cor:

① As a C_2-module $H_*(BP_{\mathbb{R}}; \mathbb{Z}_{(2)}) \cong \mathbb{Z}_{(2)}[m_1, m_2, \cdots]$

with $\gamma(m_i) = -m_i$.

[logarithm of a FGL].

② As a C_{2^n}-module.

$H_*(BP^{((C_{2^n}))}; \mathbb{Z}_{(2)}) \cong \mathbb{Z}_{(2)}[C_{2^n} \cdot m_1, C_{2^n} \cdot m_2, \cdots]$

③ Given $\{t_i^G\} \subseteq \pi_*^u BP^{((G))}$

$$\pi_*^u BP^{((G))} \cong \mathbb{Z}_{(2)}[G\cdot t_1^G, G\cdot t_2^G, \cdots]$$

if for each i.

$$\pi^u_{2(2^i-1)} BP^{((G))} \longrightarrow Q_{2(2^i-1)} \cong \mathbb{Z}_{(2)}\langle C_{2^n}\cdot m_i\rangle$$

$$t_i \longmapsto m_i - \gamma m_i$$

why?

$$Q_{2(2^i-1)} \longrightarrow \mathbb{Z}/p$$

$$m_i \longmapsto 1$$

$$\gamma m_i \longmapsto 1$$

Ker is gen. by $m_i - \gamma m_i$ equivariantly.

What we need: · How the above works?

· How to find such $\{t_i^G\}$.

Q: Can $BP_{\mathbb{R}}$ / $BP^{((G))}$ be N_∞-alg. for some N_∞-operad?

A: NO. $N_\infty \overset{\text{i.e.}}{\rightsquigarrow} E_\infty$.

Q: • What is the "best" equivariant operadic structure on $BP_{\mathbb{R}}/BP^{((G))}$?

A: I don't know.

Ref. Hill: Equivariant little disk operads.

upshot: Define "E_V-operad" for representations V.

E_σ-operad. algebra over it:

- doesn't have multiplication
- Talk about "norm" in some sense.

Q: For which V, is $BP^{"G"}$ an E_V-algebra?

Construction of $BP_{\mathbb{R}}$: Lift the Quillen idempotent on MU to $MU_{\mathbb{R}}$.

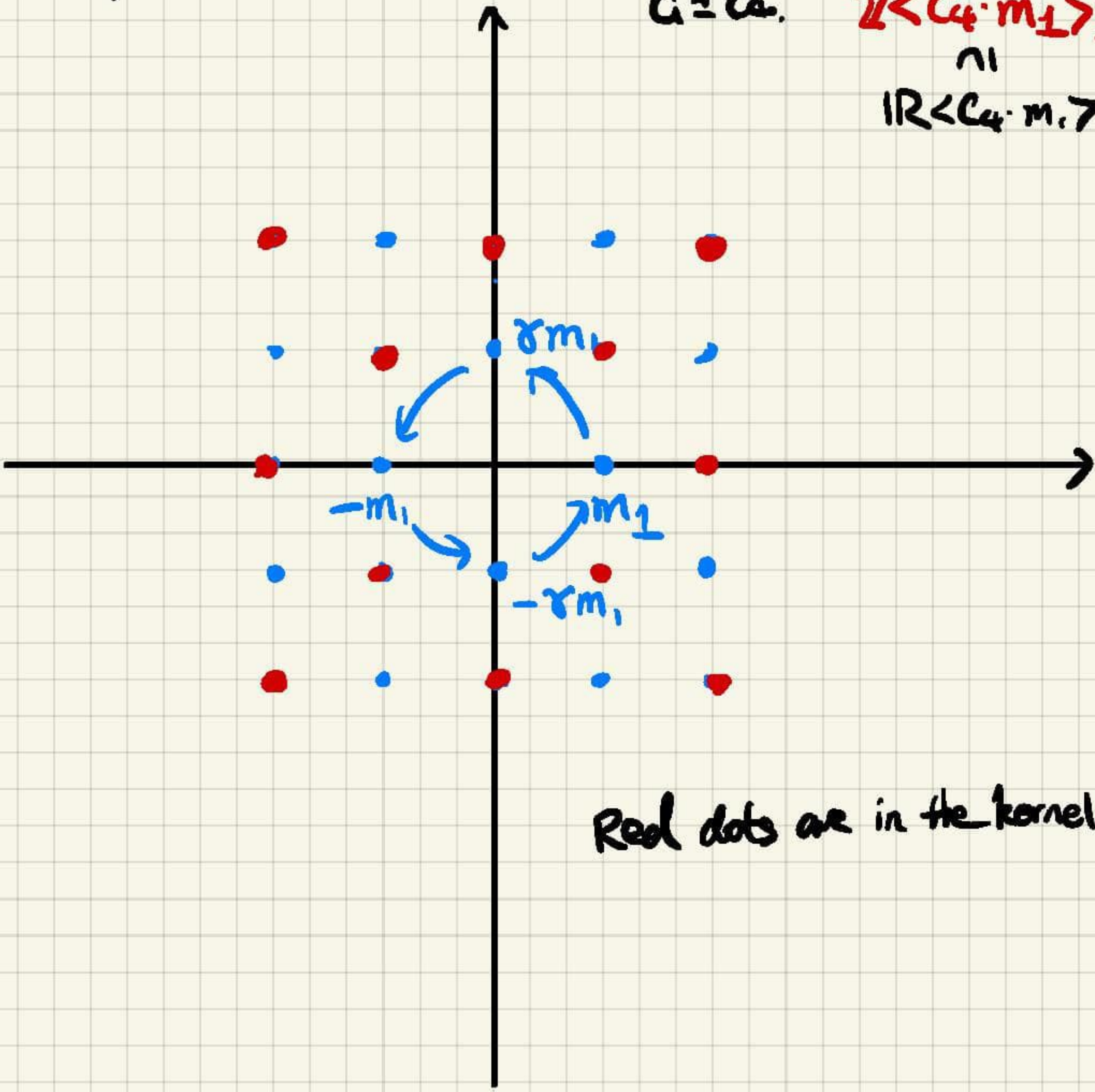
Ref. Araki. Orientations in τ-cohomology theories.

Hu-Kriz. Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence.

Question: In understanding $BP^{``C_2"}/MU^{``C_2"}$, where do we use $G = C_{2^n}$?

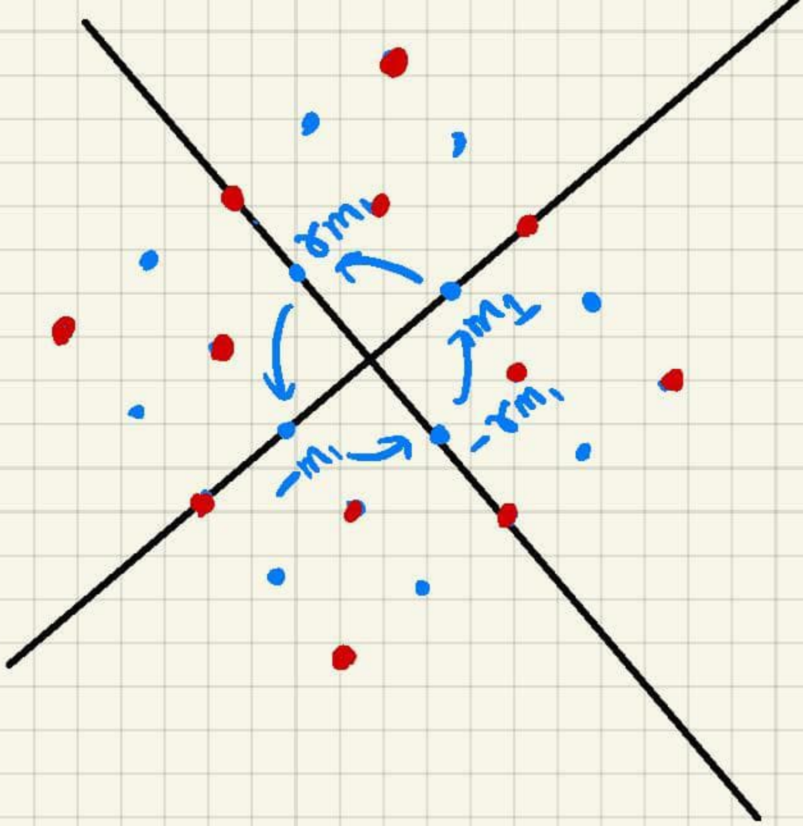
- Why $MU^{``Q_8"}/BP^{``Q_8"}$ is much harder?

$G = C_4$. $\mathbb{Z}\langle C_4 \cdot m_1 \rangle$
$\cap$
$\mathbb{R}\langle C_4 \cdot m_1 \rangle$



Red dots are in the kernel.

$$\mathbb{Z}\langle C_4 \cdot m_1 \rangle \cong Q_2 \twoheadrightarrow \mathbb{Z}/2$$
$$m_1 \longmapsto 1$$
$$\gamma m_1 \longmapsto 1$$



$$\mathrm{Ker}(\mathbb{Z}\langle C_4\cdot m_1\rangle \twoheadrightarrow \mathbb{Z}/2) \cong \mathbb{Z}\langle C_4\cdot m_1\rangle.$$

How to find these red dots in $\pi_*^{\mu} BP^{((G))}$?

i.e. how to find $G\cdot t_i^q$ s.t. $t_i^q \longmapsto m_i - \gamma m_i$ in $Q_{2r(2^i-1)}$?

· Formal Group Laws.

Ref. Ravenel Green book
[Complex cobordism and the stable homotopy groups of sph

Def. R comm. ring. A formal group law (FGL) over R.
is $F(x,y) \in R[[x,y]]$. s.t.

(1) $F(x,0) = F(0,x) = x$

(2) $F(x,y) = F(y,x)$

(3) $F(x, F(y,z)) = F(F(x,y), z)$.

Exercise: If F is a FGL$/R$, $\exists\, \iota(x) \in R[[x]]$
s.t. $F(x, \iota(x)) = 0$ $\quad \iota(x)$: inverse of F.

Ex: $R = \mathbb{Z}$ $\quad F(x,y) = x+y$ $\qquad F(x,y) = x+y+xy = (x+1)(y+1) - 1$

additive FGL. $\qquad$ multiplicative FGL.

$\updownarrow$

Topological K-theory.

Def. F, G FGLs$/R$ $\quad f(x) \in R[[x]]$.

f is a homomorphism $f: F \to G$ if

$f(F(x,y)) = G(f(x), f(y))$. f is an iso

if f is invertible. i.e. $f'(0) \in R^{\times}$.

f is a strict iso. if $f'(0) = 1$

$f(x) = a_0 x + a_1 x^2 + \cdots$, $\quad f'(0) = a_0$

Def. F FGL$/R$. a logarithm of F is a strict iso

$$f: F \longrightarrow x+y.$$

Thm. If $R \cong R \otimes \mathbb{Q}$, then every FGL$/R$ has a logarith.

$$f(x) := \int_0^x \frac{dt}{F_2(t,0)} \qquad F_2(t,0) := \frac{\partial F}{\partial y}(x,y)$$

Thm. I. There is a ring L (Lazard's ring) and a FGL F/L s.t.

$\forall$ FGL G/R $\exists \varphi : L \longrightarrow R$ s.t. $G = \varphi_* F$.

II. (Lazard) $L \cong \mathbb{Z}[x_1, x_2, \cdots]$ $|x_i| = 2i$

assume in $F(x,y)$ $|x| = |y| = -2$

ask F is homogeneous of deg -2.

Pf of I: $F(x,y) = \sum a_{ij} x^i y^j$

$$L := \mathbb{Z}[a_{ij}] \Big/ \begin{array}{c} a_{10} = a_{01} = 1 \\ \text{relations given by } F(x,0) = F(0,x) : a_{ij} = a_{ji} \\ \text{and} \\ F(x, F(y,z)) = F(F(x,y), z) \ \cdots \end{array}$$

Def. A FGL over a $\mathbb{Z}_{(p)}$-alg. is p-typical if its logarithm has the form

$$\sum_{i\geq 0} \ell_i x^{p^i} \quad \ell_0 = 1.$$

Thm (Cartier) Every FGL over a $\mathbb{Z}_{(p)}$-alg. is canonically iso. to a p-typical one.

$MU_{(p)} \simeq \bigvee \Sigma^? BP$

Cartier's thm is the algebra behind it.

Construction: F FGL/R

$f(x) \in R[\![x]\!]$ $f'(0) = 1$.

DEFINE:

$$G(x,y) := f(F(f^{-1}(x), f^{-1}(y)))$$

then $f : F \xrightarrow{\cong} G$ · iso. of FGL = source F + a power series f.

* If F is p-typical then $G(x,y)$ above might NOT be p-typical.

Lemma. $G(x,y)$ is p-typical iff

$$f^{-1}(x) = \sum_{i\geq 0}^{F} t_i x^{p^i}$$

$\sum^F$: use $F(x,y)$ instead of $x+y$ to add.

Thm There is a ring V and a p-typical FGL/V, making V the universal p-typical FGL.

$V \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ $\quad |v_i| = 2(p^i - 1)$

Cartier: F FGL/R $\quad R$ torsion-free $\mathbb{Z}_{(p)}$-alg.

$$\log_F(x) = \sum_{i\geq 0} l_i x^i$$

$$\widetilde{\log}_F(x) = \sum_{j\geq 0} l_{p^j} x^{p^j}$$

$$f(x) := \widetilde{\log}_F^{-1}(\log_F(x))$$

$$G(x,y) := f(F(f^{-1}(x), f^{-1}(y)))$$

G is p-typical, defined over $R\otimes\mathbb{Q}$

- Check coef. of G is in $R \subseteq R\otimes\mathbb{Q}$

Topology:

Def. A homotopy comm. ring spectrum R is complex orientable if $\exists x \in R^2(\mathbb{C}P^\infty)$ s.t. ← cplx orientation

$$x|_{\mathbb{C}P^1 \simeq S^2} \in R^2(S^2) \cong R^0(S^0) \text{ is a unit.}$$

Prop. I. If R is complex orientable

then $R^*(\mathbb{C}P^\infty) \cong R^*[\![x]\!]$,

$$R^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong R^*[\![x,y]\!].$$

Consider $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\otimes} \mathbb{C}P^\infty$

↑ $\otimes$ of cplx line bundles

$$\begin{aligned} R^*[\![x]\!] &\xrightarrow{\otimes^*} R^*[\![x,y]\!] \\ x &\longmapsto F(x,y) \end{aligned}$$

← is a FGL.

II. the set of cplx orientations of R

$\cong$

the set of htpy ring maps $MU \to R$.

Thm (Quillen) $MU^* \cong L$ the universal FGL

$BP^* \cong V$ the universal p-typical FGL.

Ref. Lurie. Lecture notes on chromatic homotopy theory

Prop. Let R_1, R_2 be two ring spectrum and

$MU \xrightarrow{x_1} R_1$ $\qquad$ $MU \xrightarrow{x_2} R_2$ cplx orientations.

then there is a unique power series.

$f \in (R_1 \wedge R_2)^*(\mathbb{CP}^\infty)$

f is an iso. of FGLs defined by x_1, and x_2.

Where are those t_i^G in $\pi_*^{\mathcal{U}} BP^{((G))}$?

① $\pi_*(H\mathbb{Z}_{(p)} \wedge BP) \cong \mathbb{Z}_{(p)}[m_1, m_2, \cdots]$

additive FGL $\qquad$ universal p-typical FGL.

$\sum m_i x^{p^i}$ is the logarithm of the universal p-typical FGL.

Def. In $BP^{((G))}$ t_i^G is defined as: $(G = C_{2^n})$

$\sum^{F_2} t_i^G x^{2^i}$

$$\underbrace{\overset{F_1}{BP} \curvearrowright \wedge \overset{F_2}{BP} \wedge \cdots \wedge \overset{F_{2^{n-1}}}{BP}}_{2^{n-1} \text{ copies.}}$$

t_i^G: coe. of iso. from F_1 to F_2.

$$\begin{tikzcd} F_1 \arrow[rr, rightsquigarrow] \arrow[dr, "\log_{F_1}(x)"'] & & F_2 \\ & x+y \arrow[ur, "\log^{-1}_{F_2}(x)"'] & \end{tikzcd}$$

C_{2^n}-action.

$$\log_{F_1}(x) = \sum m_i x^{2^i} \qquad \log_{F_2}(x) = \sum \gamma m_i x^{2^i}$$

$$\sum^{F_2} t_i x^{2^i} = \left(\sum \gamma m_i x^{2^i}\right)^{-1} \circ \left(\sum m_i x^{2^i}\right)$$

$$\Downarrow$$

$t_i = m_i - \gamma m_i$ modulo decomposables. $\square$

We show: $\pi_*^{u} BP^{((G))} \cong \mathbb{Z}_{(2)}[G \cdot t_1^G, G \cdot t_2^G, \cdots]$

Warning: t_i^G depends on G. $H \subseteq G$

$BP^{((H))} \longrightarrow i_H^* BP^{((G))}$ unit map

$t_i^H \longmapsto$ complicated polynomials of t_i^G.

Prop. There are elements $\bar{t}_i^{C_2} \in \pi^{C_2}_{(2^i-1)\rho_{C_2}} BP^{((G))}$

s.t. $\hat{i}_e^*(\bar{t}_i^{C_2}) = t_i^{C_2}$. $\hat{i}_e^*(\gamma \bar{t}_i^{C_2}) = \gamma t_i^{C_2}$.

It turnsout : $\pi^{C_2}_{*\rho_2} BP^{((G))} \xrightarrow{\cong} \pi^{u}_{2*} BP^{((G))}$.

- v_i are not canonically chosen.

  t_i seems to be. How?

In $BP_{\mathbb{R}}$: $t_i^{C_2}$ are coe. of $[-1]_F$.

- It only works for $p=2$.

- It is NOT Hazelwinkel nor Araki generators.

  Hazelwinkel: v_i are coe. of $[p]_F$.