The Detection Theorem

30. July 2021

The detection thm.

Review: $\cdot$ $MU^{((C_8))} := N_{C_2}^{C_8} MU_{\mathbb{R}}$.

$D \in \pi_{19\rho_8}^{C_8} MU^{((C_8))}$ $\quad D = N_2^8(\bar{t}_1^{C_8}) N_2^4(\bar{t}_2^{C_4}) N_2^8(\bar{t}_8^{C_2})$

$$D^{-1} MU^{((C_8))} \rightsquigarrow \Omega := (D^{-1} MU^{((C_8))})^{C_8}.$$

Properties of Ω: $\cdot$ $\pi_{-2}\Omega \cong 0$ The gap thm.

[Slice tower of $MU^{((C_8))}$

+

$\pi_{-2}^{C_8} \Sigma^{i\rho_8} H\underline{\mathbb{Z}} \cong 0$]

$\cdot$ $\pi_{256+*}\Omega \cong \pi_*\Omega$.

[Slice differentials + HFP Thm]

Thm (Detection) Let $\theta_j \in \pi_{2^{j+1}-2}\mathbb{S}$ be an element of Kervaire invariant 1, $j > 3$, then image of θ_j in $\pi_{2^{j+1}-2}\Omega$ is non-zero.

Ω is a ring spectrum: $MU^{((C_8))}$, $D^{-1}MU^{((C_8))}$ G-comm. rings.

$(\ )^{C_8}$ gets ring spectrum.

$\mathbb{S} \longrightarrow \Omega \rightsquigarrow$ take $\pi_{2^{j+1}-2}(-)$.

Cor(HHR) θ_j for $j \geq 7$ doesn't exist.

Questions: • Why $G = C_8$?

• Why sth. like this can be true?

* The dark force of chromatic homotopy theory!

Origin:

Ref. K–M. Groups of homotopy spheres, I.

Thm (Kervaire–Milnor) group of smooth structures on S^n.

$n \not\equiv 2 \pmod 4$

cokernel of J-homomorphism.

$$0 \longrightarrow \theta_n^{bp} \longrightarrow \theta_n \longrightarrow \pi_n/J \longrightarrow 0$$

subgp. that bounds a parallelizable mfd.

$n \equiv 2 \pmod 4$

$$0 \longrightarrow \theta_n^{bp} \longrightarrow \theta_n \longrightarrow \pi_n/J \xrightarrow{\Phi} \mathbb{Z}/2 \longrightarrow \theta_{n-1}^{bp} \longrightarrow 0$$

Kervaire invariant map.

Thm (Browder) When $n \neq 2^{j+1}-2$ Φ is zero.

If $n = 2^{j+1}-2$ Φ is non-zero iff

$h_j^2 \in \mathrm{Ext}_{\mathcal{A}}^{2, 2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ is a permanent cycle

in the $p=2$ Adams SS of $\mathbb{S}$.

The Adams spectral sequence: Compute $\pi_* \mathbb{S}_p^\wedge$.

Def. The Steenrod alg. $\mathcal{A}$ is gen. over $\mathbb{F}_2$ by

$Sq^i \quad |Sq^i| = i \quad$ Adem relation:

If $i < 2j \quad Sq^i Sq^j = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k.$

* $\mathcal{A}$ acts on $H^*(X;\mathbb{F}_2)$ for all X:

Thm (Serre) $\mathcal{A} \cong \pi_* F(H\mathbb{F}_2, H\mathbb{F}_2)$

Consider $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ very big. $\Longrightarrow \pi_* \mathbb{S}_2^\wedge$.

$\Downarrow$

$h_j \qquad |h_j| = (1, 2^j)$

Hopf inv. one problem. $\qquad Sq^{2^j}$.

$\mathcal{A}$ is a Hopf algebra: It has comul. map

$$\Delta(Sq^i) = \sum_{j=0}^{i} Sq^{i-j} \otimes Sq^j.$$

dual Steenrod algebra

Thm (Milnor) Let $\mathcal{A}_* := \mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}, \mathbb{F}_2)$

then $\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \cdots] \quad |\xi_i| = 2^i - 1$

$$\Delta(\xi_i) = \sum_{j=0}^{i} \xi_{i-j}^{2^j} \otimes \xi_j.$$

UCT: $\mathcal{A}_* \cong \pi_* H\mathbb{F}_2 \wedge H\mathbb{F}_2$.

Thm (Adams) There is a SS $E_2 \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_* \mathbb{S}_2^\wedge$.

A construction:

$X_0 = \mathbb{S}$

$X_i^! := X_i \wedge H\mathbb{F}_2$

$X_{i+1} := \mathrm{hofib}(X_i \longrightarrow X_i^!)$

$$\begin{tikzcd}
\vdots \arrow[d] & \\
X_2 \arrow[r] \arrow[d] & X_2 \wedge H\mathbb{F}_2 \\
X_1 \arrow[r] \arrow[d] & X_1 \wedge H\mathbb{F}_2 \\
\mathbb{S} = X_0 \arrow[r] & H\mathbb{F}_2
\end{tikzcd}$$

Adams SS := the SS of this tower.

Q: Why $H\mathbb{F}_2$ is special?

Replace $H\mathbb{F}_2$ by ANY homotopy ring spectrum E.

The construction works for all E, but what does it compute?

For nice enough E. "coalgebra"

$(\pi_* E, \pi_* E \wedge E)$ [Hopf algebroid, affine pre-stack]

The SS has $E_2 = \mathrm{Ext}_{(\pi_* E, \pi_* E \wedge E)}(\pi_* E, \pi_* E)$

$$\Downarrow$$

π_*(some E-completion of $\mathbb{S}$)

Our choice of $E = MU$.

Thm (Quillen) $(\pi_* MU, \pi_* MU \wedge MU) \rightsquigarrow$ FGL + strict iso.

Thm (Novikov) MU-based Adams SS

$$\Downarrow$$

$\pi_* \mathbb{S}$ without completion.

$$\begin{tikzcd} & H\mathbb{Z} \arrow[dr, "/2"] & \\ MU \arrow[ur, "p^\circ"] \arrow[rr] \arrow[rr, phantom, shift left=3ex, "\circlearrowright"] & & H\mathbb{F}_2 \end{tikzcd}$$

$$\begin{tikzcd} \text{ANSS} \arrow[r, Rightarrow] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \pi_* \mathbb{S} \arrow[d, "(-)^\wedge_2"] \\ \text{ASS} \arrow[r, Rightarrow] & \pi_* \mathbb{S}^\wedge_2 \end{tikzcd}$$

$\boxed{\begin{array}{c}\text{cplx cobordism} \\ \& \\ \text{stable htpy gps of spheres.} \\ \text{Ravenel.}\end{array}}$

Another way of thinking $\mathrm{Ext}_{(\pi_* MU, \pi_* MU \wedge MU)}$

Def. $\mathcal{M}_{FG}$ is the category of pairs (R, F)

R is a comm. ring F is a FGL over R.

$(f, \psi) : (R_1, F_1) \longrightarrow (R_2, F_2)$

$f : R_1 \longrightarrow R_2$

$\psi : F_2 \xrightarrow{\cong} f_* F_1$

Think of $\mathcal{M}_{FG}$ as a "space" and

$$\mathrm{Ext}_{(\pi_* MU,\ \pi_* MU\wedge MU)} \cong H^*(\mathcal{M}_{FG}).$$

Now, given (R, F) an action of G:

$g \in G$ $\quad g \curvearrowright R$ by ring automorphism.

choose strict iso. $g_* F \xrightarrow{\cong} F$.

with compatibility. [G-object in $\mathcal{M}_{FG}$].

Think of $G \curvearrowright (R, F)$ as a "subspace" of $\mathcal{M}_{FG}$.

This gives
$$\begin{array}{ccc} H^*(\mathcal{M}_{FG}) & \longrightarrow & H^*(G; R) \\ \cong & & \\ \mathrm{Ext}_{(\pi_* MU,\ \pi_* MU\wedge MU)} & & \end{array}$$

"Every time there is a group action on a FGL, there is a map from ANSS-E_2 to the group cohomology."

— Mike Hopkins.

It can be upgraded:

If E is a ring spectrum with G-action [at least A_∞].

$\pi_* E$ has a FGL from $MU \longrightarrow E$

then:

$$\begin{tikzcd} \text{ANSS} \arrow[r, Rightarrow] \arrow[d] \arrow[dr, phantom, "\curvearrowright"] & \pi_* \mathbb{S} \arrow[d] \\ H^*(G; \pi_* E) \arrow[r, Rightarrow, "\text{HFPSS}"] & \pi_* E^{hG} \end{tikzcd}$$

Thm (Algebraic detection)

$$\begin{tikzcd} & \pi_* \Omega \\ \text{ANSS} \arrow[r] \arrow[d] & \text{HFPSS}(D^{-1} MU^{((C_8))}) \arrow[u, Rightarrow] \\ \text{ASS} & \end{tikzcd}$$

For all $x \in \mathrm{Ext}^{2, 2^{j+1}}_{MU_*MU}(MU_*, MU_*) \leftarrow$ ANSS - E_2

s.t. $x \longmapsto h_j^2 \in \mathrm{Ext}_A^{2, 2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$

then $x \longmapsto$ non-zero in $\text{HFPSS}(D^{-1} BP^{((C_8))})$

* Why C_8?

* How to prove such a thing?

What happens in $D^{-1} MU^{((G))}$?

History: Ravenel asked in 1984:

Does $v_n^{-1} BP$ splits into $v_n^{-1} BP\langle n\rangle$?

Answer: maybe not, but true after $L_n(-)$

Hovey – Sadofsky.

Our observation:

$\bar{t}_1^{-1} BP_{\mathbb{R}}$, differentials look like

$\bar{t}_1^{-1} BP_{\mathbb{R}}\langle 1\rangle$.

$\bar{t}_n^{-1} BP_{\mathbb{R}}$. differentials look like

$\bar{t}_n^{-1} BP_{\mathbb{R}}\langle n\rangle$.

Conjecture 1. C_2-equivariantly

$L_n \bar{t}_n^{-1} BP_{\mathbb{R}}$ splits into $L_n \bar{t}_n^{-1} BP_{\mathbb{R}}\langle n\rangle$.

Conjecture 2. C_{2^n}-equivariantly

For some smart choice of D.

$L_n D^{-1} BP^{((C_{2^n}))}$ splits into $L_n D^{-1} BP^{((C_{2^n}))}\langle m\rangle$.

We know:

$$
\begin{tikzcd}
L_n D^{-1} BP^{((C_{2^n}))} \arrow[r, shift left] \arrow[d] & L_n D^{-1} BP^{((C_{2^n}))}\langle m\rangle \arrow[l, shift left, "\text{as splitting}"] \arrow[dl, bend left] \\
E_{2^{n-1}\cdot m} \circlearrowleft C_{2^n} &
\end{tikzcd}
$$

Ref. The non-existence of odd primary Arf invariant elements in stable homotopy. Ravenel 1978.

E_∞ – v.s. comm. ring.

Ref. Thm. 1.2.4. of Moduli Problems for Structured Ring Spectra by Goerss–Hopkins.

---

① Why C_8? Automorphism of FGLs.

Def. $Aut(F/R)$ is the group of strict auto. of F/R.

Example. $F = x+y$ $R = \mathbb{F}_2$.

$f \in Aut(F/R)$ if $f(x)+f(y) = f(x+y)$

$f(x) = \sum a_i x^{i+1} \Longrightarrow \sum a_i x^{i+1} + \sum a_j y^{j+1} = \sum a_i (x+y)^{i+1}$

$\Longrightarrow a_i = 0$ unless $i = 2^k - 1$.

$\Downarrow$

$\mathbb{Z}[G]$

$H^*(G;M) := \mathrm{Ext}_{\mathbb{Z}[G]}(\mathbb{Z},M)$

$\mathbb{Z}[G]^\wedge$

$H^*(G;M) := \mathrm{Ext}_{\mathbb{Z}[G]^\wedge\text{-cont}}(\mathbb{Z},M)$

$f(x) = \sum \xi_i x^{2^i}$

Given any R $\mathbb{F}_2$-alg. $\mathrm{Aut}(x+y/R)$

$\mathbb{F}_2\text{-alg}(\mathcal{A}_*, R) \cong$

$$\mathrm{Aut}(F/R) \cong \mathbb{F}_2[\xi_1, \xi_2, \cdots] \qquad |\xi_i| = 2^i - 1$$

$$\mathrm{Aut}(F/R) \cong \mathcal{A}_*$$

if $|x| = -1$

$|f(x)| = -1$.

$\Delta(\xi_i) =$ composition of power series.

$$= \sum_{j \geq 0}^{i} \xi_{i-j}^{2^j} \otimes \xi_j.$$

Adams SS $- E_2 \cong H^*(\mathrm{Aut}(F/R); \mathbb{F}_2)$.

Def. a FGL F/R R: comm. $\mathbb{F}_p$-alg.

F has height h if

$[p]_F(x) = a x^{p^h} +$ higher terms.

if $[p]_F(x) = 0$ height $= \infty$.

- height is iso. inv.
- height classifies FGLs / separably closed fields of char. p.

Thm (Hewett) Let F_h be a FGL of height h.
over $\mathbb{F}_2$, then

$C_{2^n} \subseteq \mathrm{Aut}(F_h)$ iff $h = 2^{n-1} \cdot m$
for $m > 0$.

Reminder: $BP_\star \cong \mathbb{Z}_{(2)}[t_1^{C_2}, t_2^{C_2}, \cdots]$

$t_i^{C_2}$ represents height i.

Thm (Lubin–Tate) [universal deformation]
There is a complete local ring $E_{h\star}$, and a FGL $\widehat{F}_h$ over it.
s.t. ① $E_{h\star}/m \cong \mathbb{F}_2$ and $\widehat{F}_h/m$ has height h.
② $\widehat{F}_h$ is p-typical and $t_1^{C_2}, \cdots, t_h^{C_2}$ form
a regular sequence in $E_{h\star}$.

$$\begin{array}{c} (E_{h\star}, \widehat{F}_h) \\ \big\downarrow \\ (\mathbb{F}_2, \widehat{F}_h/m) \end{array}$$

is like "universal cover of spaces".

$\mathrm{Aut}(\widehat{F}_h/m, \mathbb{F}_2) \curvearrowright (E_{h\star}, \widehat{F}_h)$

Thm (Goerss – Hopkins – Miller)

There is comm. ring spectrum E_h s.t.

- $\pi_* E_h \cong E_{h*}$ with FGL $\tilde{F}_h$.

- $\mathrm{Aut}(\hat{F}_h/m) \curvearrowright E_h$ as comm. ring maps.

$h = 4$ $\quad C_8 \curvearrowright E_4$.

Thm (Ravenel, 1978) $C_p \curvearrowright E_{p-1}$ for p odd.

$p > 3$. $E_{p-1}^{hC_p}$ detects odd primary analog of the Kervaire inv. one elements.

HHR v1. $C_8 \curvearrowright E_4$. $E_4^{hC_8}$ detects all Kervaire inv. one elements.

Problem: $E_4^{hC_8}$ is difficult to compute.

open as of today.

Turn to $MU^{((C_8))}$ $\qquad MU^{((C_8))} \longrightarrow E_4$ C_8-equivariant.

② How to prove the detection thm?

Thm (Algebraic detection)

$$\begin{tikzcd} & & \pi_*\Omega \\ \text{ANSS} \arrow[r] \arrow[d] & \text{HFPSS}(D^{-1}MU^{((C_8))}) \arrow[ru, Rightarrow] & \\ \text{ASS} & & \end{tikzcd}$$

For all $x \in \mathrm{Ext}^{2, 2^{j+1}}_{MU_*MU}(MU_*, MU_*)$ ← ANSS - E_2

s.t. $x \longmapsto h_j^2 \in \mathrm{Ext}_A^{2, 2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$

then $x \longmapsto$ non-zero in $\text{HFPSS}(D^{-1}BP^{((C_8))})$

$$\begin{tikzcd} \mathrm{Ext}^{2, 2^{j+1}}_{MU_*MU} \arrow[r] \arrow[d] & H^2(C_8; \pi_{2^{j+1}} D^{-1}MU^{((C_8))}) \arrow[d, dashed] \\ \mathrm{Ext}_A^{2, 2^{j+1}} \arrow[r, hook] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

if we can find this map to make the diagram commute

$$\mathrm{Ext}_A^{2, 2^{j+1}} \cong \mathbb{Z}/2 = \langle h_j^2 \rangle \longmapsto \frac{1}{2}$$

Original: map into $H^2(C_8; E_{4*})$

- $D \longmapsto$ unit $\Longrightarrow$

$$\begin{tikzcd} MU^{((C_8))} \arrow[r] \arrow[d] & E_4 \\ D^{-1}MU^{((C_8))} \arrow[ru, dashed] & \end{tikzcd}$$

- HHR version 1.

v_4: maps $H^2(C_8; \pi_{2\bar{\rho}+1} D^{-1} MU^{((C_8))})$ into a smaller ring.

$A := \mathbb{Z}_2[\zeta]$ ζ is a 8th root of 1.

$A_* := A[u^{\pm}]$ $|u| = 2$

$C_8 \curvearrowright A_*$ via $u \longmapsto \zeta u$.

Lubin-Tate (formal A-module)

$\exists$ FGL F/A_*

- height $(F/m) = 4$ we need $a = \zeta$

- $C_8 \curvearrowright (A_*, F)$.

[A_* is NOT π_* of something]
— Andrew Salch

F FGL.

$[n]_F$. $n \in \mathbb{Z}$.

extend: $n \in \mathbb{Z}_p$

$\mathbb{Z}_p \hookrightarrow A$

Hope to define

$\forall a \in A$

$[a]_F$ s.t. it generalizes

$[n]_F$ $n \in \mathbb{Z}_p$.

This gives: $\pi_*^u MU^{((C_8))} \longrightarrow A_*$ C_8-ring map.

- $i_e^*(D) \longmapsto$ unit in A_*.
  where the choice of D matters.

$\chi: H^2(C_8; \pi^u_{2\bar{\rho}+1} D^{-1} MU^{((C_8))}) \longrightarrow H^2(C_8; A_{2\bar{\rho}+1})$

$\pi = \zeta - 1$ uniformizer.

diagram:

$$
\begin{tikzcd}
\mathrm{Ext}^{2,2^{\bar{j}+1}}_{MU_*MU} \arrow[rr] \arrow[d] \arrow[drr, phantom, "(?)"] & & H^2\left(C_8; \pi^{\mu}_{2^{\bar{j}+1}} D^{-1} MU^{\text{``}C_8\text{''}}\right) \arrow[d, "\frac{2}{\pi}\cdot\chi"] \\
\mathrm{Ext}^{2,2^{\bar{j}+1}}_{A} \cong \mathbb{Z}/2 \arrow[d, phantom, "\cong" rotate=-90] & & H^2(C_8; A_{2^{\bar{j}+1}}) \arrow[d, phantom, "\cong" rotate=-90] \\
H^1(C_8; A_{2^{\bar{j}+1}}/(\pi)) \cong \mathbb{Z}/2 \arrow[rr, hook] \arrow[urr, phantom] \arrow[rr, bend left, "4\times\text{generator}"] & & A_{2^{\bar{j}+1}}/(8) \\
& & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}
$$

$\leftarrow \bar{j} > 3$ $C_8 \curvearrowright A_{2^{\bar{j}+1}}$ trivially.

$A_{2^{\bar{j}+1}}/(8) \xrightarrow{i} \mathbb{Q}/\mathbb{Z}$, gen $\mapsto \frac{1}{8}$.

$\mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$, $1 \longmapsto \frac{1}{2}$

If the diggram commutes we are done.

Hard: • Understand $\mathrm{Ext}^{2,2^{\bar{j}+1}}_{MU_*MU}$ [Shimomura]

$$\mathrm{Ext}^{2,2^{\bar{j}+1}}_{MU_*MU} \longrightarrow \mathrm{Ext}^{2,2^{\bar{j}+1}}_{A}$$

and

$$\mathrm{Ext}^{2,2^{\bar{j}+1}}_{MU_*MU} \longrightarrow H^2(C_8; A_{2^{\bar{j}+1}})$$

[determined by the FGL $\tilde{F}$ on A_*.

check commutativity. Formal A-modules. □

Why HFPSS is hard?

By GHM: $C_8 \curvearrowright E_{4*} = \pi_* E_4$

$$\cong$$

$C_8 \curvearrowright W(\mathbb{F}_2)[[u_1, u_2, u_3]][u^{\pm}]$

Difficulty 1: How $G \curvearrowright \pi_* E_n$ is not clear.

Difficulty 2: Differentials.