Def. A category $\mathcal{C}$ consists of

- collection of objects $X \in \mathcal{C}$
- for $X, Y \in \mathcal{C}$, set of morphisms $\mathcal{C}(X,Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X,Y)$
- identity $id_X \in \mathcal{C}(X,X)$
- composition. (unital + associative).

A functor. $F: \mathcal{C} \to \mathcal{D}$, for $X \in \mathcal{C}$, $FX \in \mathcal{D}$,
for each $f: X \to Y$ $Ff: FX \to FY$.
F should respect identities and composition.

A natural transformation between $F, G: \mathcal{C} \to \mathcal{D}$.
$\tau: F \Rightarrow G$, assigns for each $X \in \mathcal{C}$,
$\tau_X: FX \to GX$, in a natural way

$$X \xrightarrow{f} Y \qquad \begin{array}{ccc} FX & \xrightarrow{\tau_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\tau_Y]{} & GY \end{array}.$$

Ex/. Category of sets.
Category of groups

A morphism $f: X \to Y$ is an isomorphism. if there exists $g: Y \to X$, $f \circ g = id_Y$ $g \circ f = id_X$.

Adjunctions

Limits & Colimits.

Def. $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$

$F \dashv G$ (F is left adjoint of G)
if there are natural bijections
$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$
$\forall$ $X \in \mathcal{C}$ $Y \in \mathcal{D}$.

A small category is a cat whose collection of objs form a set.

Let I be a small cat, $\mathcal{C}$ be another category, the category I-shaped diagrams $\mathrm{Fun}(I,\mathcal{C})$.

I:

$\bullet$

$\bullet \to \bullet$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

$\bullet \to \bullet \to \bullet \to \bullet \to \cdots \to \bullet$

There is a diagonal functor $\mathcal{C} \xrightarrow{\Delta} \mathrm{Fun}(I,\mathcal{C})$

$X \longmapsto$ const I-shaped diagrams.

If Δ has a left adjoint, $\mathrm{Fun}(I,\mathcal{C}) \to \mathcal{C}$

$\operatorname{colim}_{I} : F \to \mathcal{C}$.

Dually, a limit is a right adjoint to such a diagonal functor.

Ex/. I:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

pushout

$F: I \to \mathcal{C}$ is an I-shaped diagram.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & \operatorname{colim} F \end{tikzcd}$$

$\operatorname{colim}_{I} F$

Def of adj:

$$\mathcal{C}(\operatorname{colim}_{I} F, W) \cong \underline{\mathrm{Fun}(I,\mathcal{C})(F, \Delta W)}.$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[drr, bend left] & & \\ Z \arrow[drr, bend right] & & W \arrow[r, "id"] \arrow[d, "id"'] & W \\ & & W & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, bend right] & \operatorname{colim} F \arrow[dr] & \\ & & W \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

Ex/. I = discrete category (no non-trivial morphisms)

colimit: coproducts

$I = \bullet \quad \bullet$

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X \amalg Y \arrow[d] & Y \arrow[l] \arrow[dl] \\ & Z & \end{tikzcd}$$

Let Top be the category of CGWH spaces

(compactly generated weak Hausdorff).

nice spaces, including all CW cplxes

w/ morphisms continuous maps btwn spaces.

- Top is bicomplete (it has all small colimits & limits).

Ex/. $X, Y, Z \in \mathrm{Top}$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \\ Z \arrow[r] & Y \sqcup_X Z \end{tikzcd}$$

$$Y \sqcup_X Z = Y \sqcup Z / f(x) \sim g(x),\ x \in X.$$

as a set

In particular, if Z is a point, f is an inclusion

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & Y/X \end{tikzcd} \leftarrow \text{cofiber of } X \hookrightarrow Y$$

make up : $*$. terminal obj of Top ($X \xrightarrow{\exists!} *$)

ϕ initial obj of Top ($\phi \xrightarrow{\exists!} X$)

The reasons why we restrict to Top_{CGWH}

①. $Y \in \mathrm{Top}_{CGWH}$

$$\mathrm{Top}(X \times Y, Z) \cong \mathrm{Top}(X, Z^Y). \leftarrow \text{Cartesian closed.}$$

$- \mapsto - \times Y$ $\qquad Z \mapsto Z^Y$

mapping space

w/ compact open topology

②. smash product associative - -

Def. Let $I = [0,1]$ be the interval.

a (left) homotopy between two maps $f, g : X \to Y$

is a map $H : X \times I \to Y$.

s.t. the diagram commutes.

$$\begin{tikzcd} X \arrow[d, "i_0"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, "H"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

$X \times I$: the cylinder over X

It is straightforward to check.

$$f_1 \simeq_l f_2, \quad g_1 \simeq_l g_2$$

$$g_1 \circ f_1 \simeq_l g_2 \circ f_2.$$

$\Rightarrow$ We can define Toph: spaces + htpy classes of maps

An isomorphism in Toph is called a <u>htpy equivalence</u>.

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \qquad g \circ f \simeq id_X, \quad f \circ g \simeq id_Y$$

<u>Make up</u>. <u>equivalence of categories.</u>

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \qquad G \circ F \cong Id_{\mathcal{C}} \quad F \circ G \cong Id_{\mathcal{D}}.$$

finite set $\simeq$ finite ordinals

<u>Def</u>. For a space X, $\pi_0 X$ is the set of path component of X.

$$\pi_0 X := \{ \text{left htpy classes } * \to X \}$$

$x \in X$.

For $n \geq 1$, $\pi_n(X, x)$ is the group of left <u>htpy classes</u> of maps $f: I^n \to X$, $f|_{\partial I^n} = x$. this is preserved in htpy

group str: $f, g: I^n \to X$ $\quad f * g$, $I^n \cong I^n \cup_{I^{n-1}} I^n \xrightarrow{f \cup g} X$. $\quad \boxed{I^n} \to \boxed{I^n | I^n}$

We say $f: X \to Y$ is a <u>weak htpy equiv</u>. if $\pi_* f$ is an isomorphism. (ROUGH).

<u>Prop</u>. Any htpy equiv is a weak htpy equiv $\square$.

<u>Def</u>. $S^n = \{ x_1^2 + \cdots + x_{n+1}^2 = 1 \} \subseteq \mathbb{R}^{n+1}$

$$D^{n+1} = \{ x_1^2 + \cdots + x_{n+1}^2 \leq 1 \} \subseteq \mathbb{R}^{n+1}.$$

S^0: 2 points. boundary $D^1 \cong I^1$.

D^0: $*$ $\quad S^{-1}$: ϕ.

<u>Def</u>. For $X \in Top$, <u>n-cell attachment</u>. is the pushout of the following

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r, "\text{attaching map}"] \arrow[d] & X \arrow[d] \\ \coprod D^n \arrow[r] & X \cup_{\coprod S^{n-1}} D^n \end{tikzcd}$$

disjoint unions of $S^{n-1} \subset D^n$

A <u>relative CW complex</u> $X \to Y$ is a countable colimit of

$$\begin{tikzcd} X = X_0 \arrow[r, "0\text{-cell}"] \arrow[drr, bend right] & X_1 \arrow[r, "1\text{-cell}"] \arrow[dr] & X_2 \arrow[r] \arrow[d] & \cdots \\ & & Y & \end{tikzcd}$$

sequential colimit.

A CW complex (absolute). $\emptyset \to Y$.

A cell complex (arbitrary attachments).

Ex/. S^n. a 0-cell and a n-cell.

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\ D^n \arrow[r] & S^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

<u>Prop</u>. For any space X, there exists a CW complex $\tilde{X}$ w/ a weak htpy equiv $\tilde{X} \to X$.

(could be made function)

<u>Prop</u> (Whitehead's Theorem). If $f: X \to Y$ is a weak homotopy equivalence, then f is a h.e.

Serre fibrations, cofibrations.

<u>Def</u>. A map $p: E \to B$ is a <u>Serre fibration</u>, if it has right lifting property for all maps $\{D^n \to D^n \times I\}_{n \geq 0}$

if
$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] & B \end{tikzcd}$$
then
$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

i.e. $D^n \times I \to B$ could be lifted to $D^n \times I \to E$ if one end of the htpy has a lift.

Ex/. A covering space $E \to B$ is a Serre fibration (lift is unique there).

FACT. A Serre fibration has the RLP against all $X \to X\times I$ if X is a CW complex.

this is an acyclic cofibration

Prop. Let $f: X \to Y$ is a Serre fibration and $y \in Y$ be a point of Y, define.

$$F_y := f^{-1}(y).$$

then there is an exact sequence for any $x \in F_y$,

$$\pi_*(F_y, x) \xrightarrow{i_*} \pi_*(X, x) \xrightarrow{f_*} \pi_*(Y, y).$$

Pf (Sketch): $\underbrace{F_y \to X \to Y}_{\mathrm{const}_y}$, so $\mathrm{im}(i_*) \subseteq \ker(f_*)$

Suppose $[\alpha] \in \ker(f_*)$ represented by $\alpha: S^{n-1} \to X$.

Since $f \circ \alpha$ is homotopic to constant map.

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ D^n \arrow[r, "F"] & Y \end{tikzcd}$$

(extend $f \circ \alpha$ to a map $F: D^n \to Y$).

Since $f(x) = y$, there is a homotopy.

$$\begin{tikzcd} & S^{n-1} \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "f"] \\ & D^n \arrow[r, "F"] \arrow[d] & Y \arrow[dd, "\mathrm{id}"] \\ D^n \arrow[r] \arrow[d] & D^n \times I \arrow[dr, "H"] & \\ * \arrow[rr] & & Y \end{tikzcd}$$

D^n is contractible so $F: D^n \to Y$ is homotopic to constant map

($H: F \simeq \mathrm{const}_y$).

$X \to Y$ is a Serre fibration, so we lift

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d] & X \arrow[dd] \\ S^{n-1} \times I \arrow[ur, dashed] \arrow[d] & \\ D^n \times I \arrow[r, "H"] & Y \end{tikzcd}$$

this is a homotopy of α to a map $\tilde{\alpha}$ which lies entirely in F_y.

(the other end of $D^n \times I$ is mapped to y)

$\tilde{\alpha} \simeq \alpha$

$\tilde{\alpha}: S^{n-1} \to F_y$, $[\alpha] = \mathrm{im}[\tilde{\alpha}]$

Def. A map $f: X \to Y$ in Top, is a cofibration, if it is a retract of a relative cell complex.

($X \to Y$, Y is formed by attaching cells to X).

A retract of a morphism $f: X \to Y$ is $g: A \to B$ which fits into

$$\begin{tikzcd} \mathrm{id}_A: & A \arrow[r] \arrow[d, "g"'] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\ \mathrm{id}_B: & B \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

(g is a retract of f)

Rmk. We have defined

- weak htpy equiv
- cofibration
- Serre fibrations.

(they are data of a model category structure)

List of properties:

① weak homotopy equivalences satisfy 2 out of 3.

i.e. $g, f, g \circ f$. if 2 of them are w.h.e. then so is the other.

② every cts map $f: X \to Y$ factors as a composite of cofibration then acyclic Serre fibration (w.h.e.).

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{cof}"'] & & Y \\ & \tilde{X} \arrow[ur, "\text{w.h.e.} + \text{Serre fib.}"'] & \end{tikzcd}$$

③ every cts map $f: X \to Y$ factors as a composite of acyclic cof followed by Serre fib.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{cof} + \mathrm{whe}"'] & & Y \\ & \tilde{Y} \arrow[ur, "\text{Serre fib.}"'] & \end{tikzcd}$$

(small object argument)

④ acyclic Serre fib are precisely those having RLP against Cof

⑤ Serre fib are precisely those w/ RLP against acyclic cof.

Def. The <u>homotopy category</u> of the model category Top, denoted by Ho(Top), is the category localized with respect to the class of weak. homotopy equivalences.

Rmk Toph. (classical htpy cat) is different

Invert all weak htpy equivalences formally.

FACT. $Ho(Top) \simeq Top_{CW}/\sim$

Ho(Top) is equivalent to the classical htpy category of CW-complexes. And this is the category we are working with.

(done after cofibrant/fibrant replacement, you can replace a space by a CW cmplx weakly htpy equiv to it, (CW complexes are cofibrant-fibrant objects).

Pointed spaces

Def. A <u>pointed space</u> (based space) is a space X with a basepoint $x \in X$. A morphism is required to respect basepoints.

$$(X, x) \xrightarrow{f} (Y, y)$$

f is cts, $f(x) = y$.

We get a category Top_* of based (CGWH) spaces.

There is an adjunction:

$$Top \underset{u.\ \text{forgetful}}{\overset{(-)_+}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} Top_*$$

i.e. $Top_*(X_+, Y) \cong Top(X, Y)$.

$$X_+ := X \amalg *$$

As Top, Top_* is also <u>bicomplete</u>:

e.g. $(X,x)\times(Y,y)\cong(X\times Y,(x,y))$.

$$(X,x)\amalg(Y,y)=X\vee Y=X\amalg Y/{*\sim *}$$

(More generally, the colimit in Top_* is computed by adding the basepoint to the diagram)

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \quad \underset{\rightsquigarrow}{\text{compute colim in Top}}. \quad X\vee Y = \begin{tikzcd} * \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X\vee Y \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

In fact, Top_* is an example of <u>under category</u>:

obj: $* \to X$

morphism. $\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$

<u>Def</u>. Let X, Y be pointed spaces, the <u>smash product</u>. is

$$X\wedge Y := X\times Y/X\vee Y.$$

There is an adjunction

based at $\mathrm{const}_z: Y\to Z$

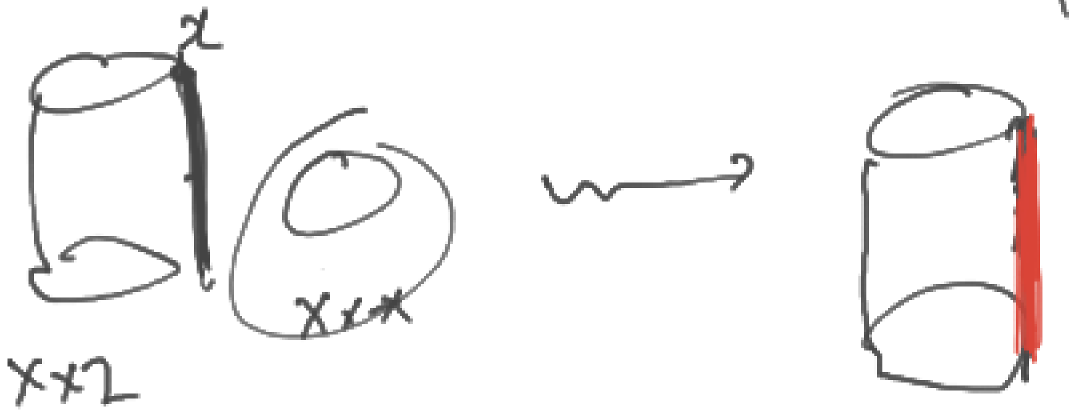
$$\mathrm{Top}_*(X\wedge Y,Z)\cong\mathrm{Top}_*(X,\mathrm{Top}_*(Y,Z)).$$

($-\wedge Y$ is left adjoint to the mapping space functor).

(this is analogous to $\mathrm{Top}(X\times Y,Z)\cong\mathrm{Top}(X,Z^Y)$)

<u>Def</u>. The <u>reduced cylinder</u> over $X\in\mathrm{Top}_*$,

is $\underline{X\wedge I_+} = X\times I_+/X\vee I_+$



$X\times I/x\times I$.

A based left homotopy between two based $f, g : (X,x) \to (Y,y)$.

is a diagram

$$\begin{tikzcd}
(X,x) \arrow[d] \arrow[dr, "f"] & \\
X \wedge I_+ \arrow[r, "H"] & (Y,y) \\
(X,x) \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

Ex/. Given a morphism $f : X \to Y$ in Top_*,

the <u>fiber</u> of f is the pullback

$$\begin{tikzcd}
fib(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
* \arrow[r] & Y
\end{tikzcd}$$

the <u>cofiber</u> of f is the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\
* \arrow[r] & cofib(f) \cong Y/X
\end{tikzcd}$$

We can define w.h.e, cofibration, fibrations in Top_*.

$\Longrightarrow$ homotopy category of pointed topological spaces. $Ho(Top_*)$

$\simeq$

classical htpy cat of <u>based CW complexes</u>.

(CW complex + 0-cell as a basepoint)

Finally, $(Top, \times, *)$ and $(Top_*, \wedge, S^0)$ are <u>symmetric monoidal categories</u>.

Now we move into htpy categories. $Ho(Top_*)$.

We want to pass constructions/functors to htpy categories.

e.g. $$\begin{array}{ccc} Top_*(X \wedge Y, Z) & \cong & Top_*(X, Map_*(Y,Z)). \\ \cap & & \cap \\ Top(X \wedge Y, Z) & & Top(X, Map_*(Y,Z)) \end{array}$$

Take π_0 (path connected component).

$$\pi_0 Top_*(X \wedge Y, Z) \cong \pi_0 Top_*(X, Top_*(Y,Z))$$

$\Longrightarrow$ $[X \wedge Y, Z] \cong [X, Top_*(Y,Z)]$ $\quad$ ($[X,Y]$ is $Ho(Top_*)(X,Y)$)

(2). pushouts and pullbacks (dual of pushout) are not well-behaved

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \\ D^n & \end{tikzcd} \quad \simeq \quad \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

equivalent in $\mathrm{Fun}(\lrcorner\,;\ \mathrm{Ho}(\mathrm{Top}_*))$.

functor category will have w.e, cof, fib

they will also have htpy cats.

(D^n is contractible)

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd} \qquad \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

$$S^n \not\simeq *$$

Recipe to resolve this : homotopy limit/colimit.

$$\mathrm{colim} \dashv \Delta_I \qquad \mathrm{Top}_* \underset{\mathrm{colim}}{\overset{\Delta}{\rightleftarrows}} \mathrm{Fun}(I, \mathrm{Top}_*).$$

derived functors

$$\mathrm{hocolim} \dashv \Delta_I \qquad \mathrm{Ho}(\mathrm{Top}_*) \underset{\mathrm{hocolim}}{\overset{\Delta}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Fun}(I, \mathrm{Top}_*))$$

$F \dashv G$ F left adjoint G.

Applications of htpy colimits / limits.

- htpy cofiber, htpy fiber, (co)fib sequences.

Recall that in Top_*: $f: X \to Y$ has cofiber:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ * \arrow[r] & \mathrm{Cofib}(f) \end{tikzcd}$$

Step 1. Mapping cylinder of f,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ X \wedge I_+ \arrow[r] & \mathrm{Cyl}(f) \end{tikzcd}$$

cylinder obj

$X \wedge I_+$ 

$\cup_f Y$ ⇝ 

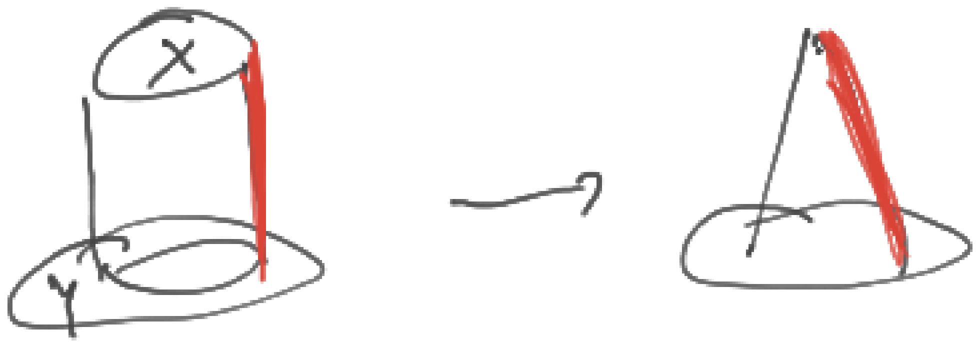
Essentially: $X \xrightarrow{f} Y$, $X \overset{\text{cofib}}{\hookrightarrow} \mathrm{Cyl}(f) \xrightarrow{\text{w.h.e.}} Y$

(if X is a CW cpx)

Step 2 Make a cofiber

$$\begin{tikzcd}
X \arrow[r, hook, "\text{including } X \text{ to top}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Cyl(f) \arrow[d] \\
* \arrow[r] & Cone(f)
\end{tikzcd}$$

Cone(f).



Put tgt:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[dd] \\
X \wedge I_+ \arrow[dr, Rightarrow] & \\
X \arrow[r] \arrow[d] & X \wedge I_+ \\
* \arrow[r] & Cone(f)
\end{tikzcd}$$

This construction of homotopy cofiber respect weak. htpy equivalences now.

$$\begin{tikzcd}
X \arrow[r, "cof", "f_1"'] \arrow[d, "\sim"'] & Y_1 \arrow[r] \arrow[d, "\sim"] & hocofib(f_1) \cong Cone(f_1) \arrow[d, "\sim"] \\
X_2 \arrow[r, "f_2", "cof"'] & Y_2 \arrow[r] & hocofib(f_2) \cong Cone(f_2)
\end{tikzcd}$$

$\Rightarrow$ If given a map $f: X \to Y$.

we can always replace it by any factorization

$$f: X \xrightarrow[cof]{} \tilde{X} \xrightarrow[w.h.e.]{} Y$$

hocofib(f) is well-defined as the cofiber of $(X \to \tilde{X})$.

Derive similarly. as the proof of exactness of

$$\pi_*(F_y) \to \pi_*(X) \to \pi_*(Y).$$

① . htpy fiber sequence,

Suppose $X, Y \in Top_*$, and $f: X \to Y$,

there is a long exact sequence for any $A \in Top_*$.

$$[A,\Omega X] \to [A,\Omega Y] \to [A, hofib(f)] \to [A, X] \xrightarrow{f_*} [A, Y]$$

$$\uparrow$$

$\Omega Y := Top_*(S^1, Y)$.

$$[A, \Omega hofib(f)]$$

$$\uparrow$$

($[X, Y]$ is based htpy classes of maps from X to Y)

$$[A, \Omega^2 Y]$$

$$\uparrow$$

$$\vdots$$

($\Omega Y \sim hofib(hofib(f) \to X)$)

$S^n \wedge S^0 = S^n \quad \Omega^n Y = Top_*(S^n, Y)$

$A = S^0$. $[S^0, \Omega^n Y] \cong \pi_n Y \cong [S^n, Y]$.

$\Rightarrow$ LES of htpy groups.

$$\cdots \to \pi_3(Y) \to \pi_2(hofib(f)) \to \pi_2(X) \to \pi_2(Y) \to \pi_1(hofib(f)) \to \pi_1(X) \to \pi_1(Y) \to *$$

② . htpy cofiber sequences.

$$X \xrightarrow{f} Y \to hocofib(f) \to \Sigma X \to \Sigma Y \to \cdots$$

$(\Sigma X \cong S^1 \wedge X)$

induces long exact sequence

$$\cdots \to [\Sigma Y, A] \to [\Sigma X, A] \to [hocofib(f), A] \to [Y, A] \to [X, A]$$