Equivariant Spectra as Presheaves

This notes is adapted from Jonathan Rubin's notes from IWoAT2019

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Jul 23, 2021

Suppose G is a finite group, and let $\mathbf{Orb}_G$ be the orbit category of G. By Elmendorf's theorem (cf. Talk 2.2), there is an equivalence between the homotopy theory of G-spaces, and the homotopy theory of topological presheaves over $\mathbf{Orb}_G$. In this note, we introduce Mackey functors and explain Guillou and May's version of Elmendorf's theorem for G-spectra. This result gives an algebraic perspective on equivariant spectra.

1 Symmetric Monoidal Categories

Definition 1.1. A symmetric monoidal category $(\mathscr{C}, \otimes, I)$ is category $\mathscr{C}$ equipped with $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and an unit object $I \in \mathscr{C}$ such that $\otimes$ is associative, unital and commmutative up to coherent natural isomorphisms α, β, λ and ρ. That is, $\mathscr{C}$ is also equipped with an associator

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C,$$

a commutator

$$\beta_{A,B} : A \otimes B \cong B \otimes A,$$

a left unitor $\lambda_A : I \otimes A \to A$ and a right unitor $\rho_A : A \otimes I \cong A$ such that they are all naturall isomorphisms and the following diagrams commute.

$$\begin{array}{ccccc} A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha} & ((A \otimes B) \otimes C) \otimes D \\ \downarrow{\scriptstyle 1\otimes\alpha} & & & & \uparrow{\scriptstyle \alpha\otimes 1} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow[\alpha]{\qquad\qquad} & & (A \otimes (B \otimes C)) \otimes D \end{array}$$

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \xrightarrow{\alpha} & (A \otimes B) \otimes C & \xrightarrow{\gamma} & C \otimes (A \otimes B) \\ \downarrow{\scriptstyle 1\otimes\gamma} & & & & \downarrow{\scriptstyle \alpha} \\ A \otimes (C \otimes B) & \xrightarrow[\alpha]{} & (A \otimes C) \otimes B & \xrightarrow[\gamma\otimes 1]{} & (C \otimes A) \otimes B \end{array}$$

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha} & A \otimes (I \otimes B) \\ & {\scriptstyle \rho \otimes 1} \searrow \quad \swarrow {\scriptstyle 1 \otimes \lambda} & \\ & A \otimes B & \end{array}$$

and

$$\begin{array}{ccc} & B \otimes A & \\ & {\scriptstyle \beta} \nearrow \quad \searrow {\scriptstyle \beta} & \\ A \otimes B & \underset{id}{=\!=\!=\!=} & A \otimes B \end{array}$$

Example 1.2. $(\mathbf{Set}, \times, \{*\})$

Example 1.3. If R is commutative, then $(R\text{−mod}, \otimes, R)$ is symmetric monoidal.

Example 1.4. $(\mathbf{Sp}, \wedge, S^0)$ is a symmetric monoidal category where $\mathbf{Sp}$ is category of symmetric spectra, orthogonal spectra or EKMM.

Theorem 1.5. *The classifying space of a symmetric monoidal category is an E_∞ space, so its group completion is an infinite loop space.*

Definition 1.6. A permutative category is a unital symmetric monoidal category where the associator α is the identity.

Theorem 1.7. *Every symmetric monoidal category is equivalent to a permutative category.*

2 Enriched Categories

Let $(\mathscr{V}, \otimes, I)$ be a monoidal category.

Definition 2.1. A small $\mathscr{V}$-enriched category $\mathscr{C}$ consists of

1. a class $\mathrm{Obj}(\mathscr{C})$ of objects of $\mathscr{C}$;
2. an object $\mathscr{C}(a, b) \in \mathscr{V}$ for each pair $(a, b) \in \mathrm{Obj}(\mathscr{C}) \times \mathrm{Obj}(\mathscr{C})$;
3. for each ordered triple (a, b, c) of objects of $\mathscr{C}$ a morphism

   $$\circ_{a,b,c} : \mathscr{C}(b, c) \otimes \mathscr{C}(a, b) \to \mathscr{C}(a, c)$$

   in $\mathscr{V}$;
4. for each object $a \in \mathrm{Obj}(\mathscr{C})$ a morphism $j_a : I \to \mathscr{C}(a, a)$;
5. such that the following diagrams commute

$$\begin{array}{ccccc} (\mathscr{C}(c,d) \otimes \mathscr{C}(b,c)) \otimes \mathscr{C}(a,b) & & \xrightarrow{\alpha} & & \mathscr{C}(c,d) \otimes (\mathscr{C}(b,c) \otimes \mathscr{C}(a,b)) \\ \downarrow {\scriptstyle \circ_{b,c,d} \otimes 1} & & & & \downarrow {\scriptstyle 1 \otimes \circ_{a,b,c}} \\ \mathscr{C}(b,d) \otimes \mathscr{C}(a,b) & \xrightarrow{\circ_{a,b,d}} & \mathscr{C}(a,d) & \xleftarrow{\circ_{a,c,d}} & \mathscr{C}(c,d) \otimes \mathscr{C}(a,c) \end{array}$$

$$\begin{array}{ccccc} \mathscr{C}(b,b)\otimes\mathscr{C}(a,b) & \xrightarrow{\circ_{a,b,b}} & \mathscr{C}(a,b) & \xleftarrow{\circ_{a,a,b}} & \mathscr{C}(a,b)\otimes\mathscr{C}(a,a) \\ \uparrow{\scriptstyle j_b\otimes 1} & \nearrow_{\lambda} & & {}_{\rho}\nwarrow & \uparrow{\scriptstyle 1\otimes j_a} \\ I\otimes\mathscr{C}(a,b) & & & & \mathscr{C}(a,b)\otimes I \end{array}$$

Example 2.2. **Set**-enriched category are precisely the usual categories.

Example 2.3. Additive categories are **Ab**-enriched categories.

Example 2.4. **Top** can be regarded as a **Top**-enriched category.

Example 2.5. With a suitable model we can also make the category of spectra **Sp** enriched in itself.

Definition 2.6. A strict 2-category is a category enriched over **Cat** where the monoidal structure in **Cat** is given by the product of categories.

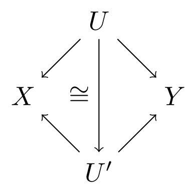
3 Spectral Mackey Functors

Elmendorf's theorem suggests that G-spectra are categorifications of Mackey functors. We now present a theorem that gives substance to this idea.

Regard a Mackey functor as an additive functor $\underline{M} : \mathscr{B}_G^{\mathrm{op}} \to \mathbf{Ab}$ from the Burnside category to the category of abelian groups. Every part of this definition has a higher algebraic counterpart. The idea is to replace **Ab** with the category **Sp** of nonequivariant spectra, and to enhance $\mathscr{B}_G$ to a spectrally enriched category $\mathscr{B}_{G,sp}$ that models the full, spectral subcategory of $G\mathbf{Sp}$ spanned by suspensions of finite pointed G-sets. We follow Guillou and May's model categorical treatment [4], where $\mathscr{B}_{G,sp}$ is denoted $G\mathscr{A}$, but Barwick has proven a similar theorem using ∞-categories [1]. An important antecedent to these results appears in earlier work of Schwede and Shipley [9, Example 3.4.(i)]. They show that $G\mathbf{Sp}$ is equivalent to the category of spectral presheaves over the full subcategory of $G\mathbf{Sp}$ spanned by $\{\Sigma_+^\infty G/H \mid H \subset G\}$. A related result for presentable stable ∞-categories is given in [7, Proposition 1.4.4.9].

Remark 3.1. The difference between [9] and the work in [1] and [4] is that the latter papers construct the spectral Burnside category without reference to equivariant homotopy theory. Barwick works over a natural ∞-categorical lift of the Lindner category $\mathscr{B}_G^+$, which he does not group complete. As we explain below, Guillou and May work over a precise spectral analogue to $\mathscr{B}_G$, constructed by homotopy group completing a 2-categorical lift of $\mathscr{B}_G^+$.

Recall that the Lindner 1-category $\mathscr{B}_G^+$ has finite G-sets $X, Y, Z, \dots$ as objects, and that a morphism from X to Y in $\mathscr{B}_G^+$ is an isomorphism class of a span $X \leftarrow U \to Y$ of finite G-sets. The first step in constructing $\mathscr{B}_{G,sp}$ is to remember the isomorphisms



between different representatives of morphisms in $\mathscr{B}_G^+$. It will be technically convenient to think of a span $X \leftarrow U \rightarrow Y$ as a single morphism $U \rightarrow Y \times X$, and to consider G-actions on the finite sets $\varnothing, \{1\}, \{1,2\}, \{1,2,3\}, \ldots$ only. Among other things, this cuts the proper class of finite G-sets down to a countable set, and it makes the disjoint union and cartesian product of G-sets strictly associative and unital. *Henceforth, we understand all finite G-sets to be of this form.*

Definition 3.2. For any finite G-set A, let $G\mathscr{E}(A)$ be the category of finite G-sets and G-isomorphisms over A. This is a strictly associative and unital symmetric monoidal category (also called a *permutative category*) under disjoint union.

Recall that a *bicategory* $\mathscr{C}$ is a category weakly enriched in 1-categories. More explicitly, a bicategory consists of a class of objects $\mathrm{Ob}(\mathscr{C})$, hom 1-categories $\mathscr{C}(X,Y)$, composition functors $\circ : \mathscr{C}(Y,Z) \times \mathscr{C}(X,Y) \rightarrow \mathscr{C}(X,Z)$, and identities $\mathrm{id} : * \rightarrow \mathscr{C}(X,X)$ such that the usual associative and unital laws hold up to coherent natural isomorphism.

Definition 3.3. Let $G\mathscr{E}$ be the bicategory whose objects are finite G-sets, and whose hom 1-categories are $G\mathscr{E}(X,Y) = G\mathscr{E}(Y \times X)$. Composition corresponds to the pullback of spans, and the diagonal $\Delta : X \rightarrow X \times X$ is the identity at X.

The bicategory $G\mathscr{E}$ is very nearly a strict 2-category. Composition is strictly associative, and one of the unit laws holds strictly. We can make the other unit law strict by "whiskering" on new identity elements. To be precise, if $\mathscr{C}$ is a 1-category with basepoint $c \in \mathscr{C}$, then the whiskered category $\mathscr{C}'$ has:

1. objects $\mathrm{Ob}(\mathscr{C}') = \mathrm{Ob}(\mathscr{C}) \sqcup \{*\}$, and
2. hom sets $\mathscr{C}'(x,y) = \mathscr{C}(\varepsilon(x), \varepsilon(y))$, where $\varepsilon : \mathrm{Ob}(\mathscr{C}') \rightarrow \mathrm{Ob}(\mathscr{C})$ is the identity map on $\mathrm{Ob}(\mathscr{C})$ and sends $*$ to the basepoint $c \in \mathscr{C}$.

Compositions and identities in $\mathscr{C}'$ are inherited from $\mathscr{C}$, and $w = \mathrm{id}_c \in \mathscr{C}'(*, c)$ is a canonical "whisker isomorphism" between $*$ and c in $\mathscr{C}'$. Thus, the inclusion $\mathscr{C} \hookrightarrow \mathscr{C}'$ is an equivalence of categories. We think of $\mathscr{C}'$ as a categorical analogue to the whiskering $X' = X \vee [0,1]$ of a based space X, but we warn the reader that the classifying space $B(\mathscr{C}')$ is not homeomorphic to $(B\mathscr{C})'$.

Definition 3.4. Let $\mathscr{B}^+_{G,2}$ be the strict 2-category whose objects are finite G-sets $X, Y, Z, \ldots$, and whose hom 1-categories are

$$\mathscr{B}^+_{G,2}(X,Y) = \begin{cases} G\mathscr{E}(X,X)' & \text{if } X = Y \text{ and } |X| > 1 \\ G\mathscr{E}(X,Y) & \text{otherwise} \end{cases}.$$

Here we regard $\Delta : X \to X \times X$ as the baspoint of $G\mathscr{E}(X,X)$.

The 2-category $\mathscr{B}^+_{G,2}$ is strictly associative and unital because the whisker isomorphisms provide room to "hang" the bicategorical unit isomorphisms of $G\mathscr{E}$. We refer the reader to [4, §5] for details, where $\mathscr{B}^+_{G,2}$ is denoted $G\mathscr{E}'$.

The homs of $\mathscr{B}^+_{G,2}$ are still permutative categories, and they should be thought of as commutative monoids up to coherent homotopy. It remains to homotopy group complete them. We can do considerably better. Given any permutative category $\mathscr{C}$, there is a connective spectrum $\mathbb{K}\mathscr{C}$ such that $\Omega^\infty\mathbb{K}\mathscr{C}$ is a group completion of the classifying space $B\mathscr{C}$ (cf. [10] and [8]). The basic idea in [10] is to construct the levels of $\mathbb{K}\mathscr{C}$ using a homotopical version of the iterated classifying space construction for topological abelian groups, but nailing down the details is nontrivial. Moreover, the classical versions of $\mathbb{K}\mathscr{C}$ will not suffice for the problem at hand, because producing an honest spectral category $\mathscr{B}_{G,sp}$ from $\mathscr{B}^+_{G,2}$ requires a construction with more precise multiplicative properties, and proving that spectral Mackey functors over $\mathscr{B}_{G,sp}$ are equivalent to G-spectra requires even more compatibilities.

Guillou, May, Merling, and Osorno have developed an "equivariant infinite loop space machine" $\mathbb{K}_G$ with all of the necessary properties in [5] and subsequent work. When $G = e$, the machine $\mathbb{K} = \mathbb{K}_e$

1. sends a permutative category $\mathscr{C}$ to a connective spectrum $\mathbb{K}\mathscr{C}$ whose 0-space is the group completion of $B\mathscr{C}$, and
2. sends multilinear maps between permutative categories to multilinear maps between spectra.

Thus, applying $\mathbb{K}$ to the hom categories of $\mathscr{B}^+_{G,2}$ produces a spectral category.

Definition 3.5. Let $\mathscr{B}_{G,sp}$ be the spectral category whose objects are finite G-sets $X, Y, Z, \ldots$, and whose hom spectra are $\mathscr{B}_{G,sp}(X,Y) = \mathbb{K}_G(\mathscr{B}^+_{G,2}(X,Y))$. A *spectral Mackey functor* is a contravariant spectral functor from $\mathscr{B}_{G,sp}$ to $\mathbf{Sp}$. We write $\mathbf{Mack}_{sp}(G)$ for the category of spectral Mackey functors for G.

Using further properties of $\mathbb{K}_G$, Guillou and May prove that the homotopy theory of spectral G-Mackey functors and the homotopy theory of G-spectra are equivalent.

Theorem 3.6 ([4, Theorem 0.1])**.** *There is a zig-zag of Quillen equivalences connecting* $\mathbf{Mack}_{sp}(G)$ *to the category* $G\mathbf{Sp}$ *of orthogonal G-spectra.*

We briefly indicate some ingredients in the proof. Just as the orbits G/H generate the homotopy theory of $G\mathbf{Top}$, the suspension spectra $\Sigma^\infty_+ G/H$ generate the homotopy theory of $G\mathbf{Sp}$. Let $G\mathscr{D}$ denote the full, spectral subcategory of $G\mathbf{Sp}$ spanned by (bifibrant approximations of) the suspension spectra of finite pointed G-sets. Then the category of spectral contravariant functors from $G\mathscr{D}$ to $\mathbf{Sp}$ is Quillen equivalent to $G\mathbf{Sp}$. This is essentially Schwede and Shipley's theorem [9]. The rest of the proof boils down to showing that $G\mathscr{D}$ is suitably equivalent to the spectral category $\mathscr{B}_{G,sp}$.

By a non-group complete version of the tom Dieck splitting, the category $\mathscr{B}^+_{G,2}(X,Y)$ is a model for the G-fixed points of $\mathbb{P}_G(Y\times X)_+$, the free G-E_∞ algebra on the based G-set $(Y\times X)_+$. Therefore $\mathscr{B}_{G,sp}(X,Y)\simeq\mathbb{K}(\mathbb{P}_G(Y\times X)^G_+)$, and compatibility relations between $\mathbb{K}$ and $\mathbb{K}_G$ imply that $\mathbb{K}(\mathbb{P}_G(Y\times X)^G_+)\simeq\mathbb{K}_G(\mathbb{P}_G(Y\times X)_+)^G$. From here, the equivariant Barratt-Priddy-Quillen theorem gives $\mathbb{K}_G(\mathbb{P}_G(Y\times X)_+)^G\simeq(\Sigma^\infty_+(Y\times X))^G$, and by duality,

$$(\Sigma^\infty_+(Y\times X))^G\cong(\Sigma^\infty_+Y\wedge\Sigma^\infty_+X)^G\simeq F_G(\Sigma^\infty_+X,\Sigma^\infty_+Y)^G\simeq F^G(\Sigma^\infty_+X,\Sigma^\infty_+Y).$$

Therefore $\mathscr{B}_{G,sp}(X,Y)\simeq G\mathscr{D}(\Sigma^\infty_+X,\Sigma^\infty_+Y)$.

Ignoring (co)fibrancy issues, the rest of the proof boils down to checking that these equivalences define an equivalence $\mathscr{B}_{G,sp}\simeq G\mathscr{D}$ of spectral categories, and that they induce an equivalence $\mathbf{Fun}(\mathscr{B}^{\mathrm{op}}_{G,sp},\mathbf{Sp})\simeq\mathbf{Fun}(G\mathscr{D}^{\mathrm{op}},\mathbf{Sp})$ on the level of homotopy theories.

Example 3.7. For any finite G-set X, the representable spectral Mackey functor $\mathscr{B}_{G,sp}(-,X)\in\mathbf{Mack}_{sp}(G)$ and the suspension G-spectrum $\Sigma^\infty_+X\in G\mathbf{Sp}$ correspond under the equivalences of Theorem 3.6 (cf. [4, §2.5]). In particular, $\mathscr{B}_{G,sp}(-,G/G)$ correponds to the sphere G-spectrum.

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