Outline

- the category of orthogonal G-spectra Sp_G^O
  - ↳ tensored & cotensored over Top^G
  - ↳ suspension spectra. (geo fixed point comm w/ Σ).
  - ↳ SW^G fully-faithful embedding into Sp_G^O
  - ↳ equivariant stable homotopy groups & π_*-iso
  - ↳ symm mon (smash prod)
- derived fixed points (categorical fixed points)
  - ↳ not comm w/ Σ!
  - ↳ as right adjs to the trivial functor.
- tom Dieck splitting: characterize the categorical fixed points of $\Sigma^\infty X$.
- Adams iso.
- Norm: multiplicative transfer.

defn of Sp_G^O

indexing category $\mathcal{O}$: obj fin dim real inner prod spaces.

$$\text{mor: } \mathcal{O}(V,W) = \begin{cases} * & \dim W < \dim V \\ \mathcal{L}(V,W)_+ & \dim W = \dim V \\ \underset{\phi: V \hookrightarrow W}{\bigvee} S^{W-\phi(V)} & \dim W > \dim V \end{cases}$$

Defn: An orthogonal G-spectrum is a based cts functor

$$\mathcal{O} \to Top_*^G$$

we denote the cat of orthogonal G-spec as Sp_G^O

unpack:
- assign every V a G space $X(V)$
- for pair (V,W), have equivariant str map

$$\mathcal{O}(V,W) \wedge X(V) \to X(W)$$

combining $\sigma_{V,W}: S^{W-V} \wedge X(V) \to X(W)$ for every $V \hookrightarrow W$

$O(V)$ action on $X(V)$

integer graded version. $V \cong \mathbb{R}^n$ for some n.

Defn: An orthogonal G-sp X is determined by.

- a based space X_n w/ a based $O(n) \times G$ action for each $n \geq 0$
- a based G-eq structure map $\sigma_n : X_n \wedge S^1 \to X_{n+1}$ for each $n \geq 0$
- for all $m, n \geq 0$, the iterated structure map

$$S^m \wedge X_n \xrightarrow{S^{m-1} \wedge \sigma_n} S^{m-1} \wedge X_{n+1} \to \cdots \to X_{n+m}$$

is $O(m) \times O(n)$-equivariant.

Rmk). not the same as in HHR.

$\mathcal{O}_G$ obj: fin dim G-repre

mor: $\mathcal{O}_G(V, W) = O(V, W)$ plus a G-action by conj.

Fact: for V a G-representation

define $X(V) := \mathcal{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$

"So can think of X is representation graded"

Sp_G^{O} tensored & cotensored over Top^G

A a G-space $X \in Sp_G^{O}$

define $A \wedge X(V) := A \wedge X(V)$ G acts diagonally.

$Map(A, X)(V) := Map(A, X(V))$ G acts by conj.

ex) $\Sigma^V X := S^V \wedge X$ $\Omega^V X := Map(S^V, X)$.

Suspension spectrum

$$\Sigma^\infty A(V) := S^V \wedge A$$

SW^G fully-faithful embedding into Sp_G^{O}

$$[\Sigma^\infty X, \Sigma^\infty Y]_G \cong \operatorname*{colim}_{V \in \mathcal{U}} [\Sigma^V X, \Sigma^V Y]_G = \{X, Y\}_G.$$

for each X, Y fin G CW.

equivariant stable htpy groups of orthogonal G-spectra.

H closed subgroup of G. $X \in Sp_G^O$ Cary's note

H-eq 0-th stable htpy group of X:

$$\pi_0^H(X) := \operatorname*{colim}_{V \in \mathcal{U}} [S^V, X(V)]_H$$

$\boxed{\text{non eq}}$

$$\begin{tikzcd}
\text{Orth sp} \arrow[d] \arrow[dr, "\operatorname{colim} \pi_{k+n}(X_k)"] \arrow[dr, phantom, shift right=4, "\circlearrowleft"] & \\
\text{Stable htpy cat} \arrow[r, "{[\mathbb{S}, X]_*}"'] & \text{GrAb}
\end{tikzcd}$$

k positive integer.

$$\pi_k^H(X) := \operatorname*{colim}_V [S^{V \oplus \mathbb{R}^k}, X(V)]_H$$

$$\pi_{-k}^H(X) := \operatorname*{colim}_V [S^V, X(V \oplus \mathbb{R}^k)]_H$$

π_*-iso (weak eq). strict model str.

$X \xrightarrow{f} Y \in Sp_G^O$ is a π_*-iso if

$$\pi_n^H(f) : \pi_n^H(X) \to \pi_n^H(Y)$$

is iso for all closed $H \leq G$ and all integer n.

smash prod on Sp_G^O

smash prod of non eq + diagonal G-action.

defn: $X, Y \in Sp_G^O$ $X \wedge Y \in Sp_G^O$ the n-th level is the coequalizer

$$\bigvee_{p+1+q=n} O(n)_+ \underset{O(p)\times 1\times O(q)}{\wedge} X_p \wedge S^1 \wedge Y_q) \underset{X_p \wedge \alpha_q^Y \circ X_p \wedge \mathrm{braid}_{S^1, Y_q}}{\overset{\alpha_p^X \wedge Y_q}{\rightrightarrows}} (\bigvee_{p+q=n} O(n)_+ \underset{O(p)\times O(q)}{\wedge} X_p \wedge Y_q) \to (X \wedge Y)_n$$

$$\exists! \quad \to Z.$$

naturally extends to a functor $Sp_G^O \times Sp_G^O \to Sp_G^O$

this makes $(Sp_G^O, \wedge, \mathbb{S})$ a symm mon category.

(closed) has internal hom.

geo fixed point comm w/ suspension

$\Sigma^\infty A(U) := S^U \wedge A$

recall $\phi^G = (\tilde{E}P \wedge X)^G$

$\Sigma_G^\infty: G\,Top \to G\,Sp.$ $\qquad \phi^G(\Sigma_G^\infty A) \cong \Sigma^\infty(A^G)$

derived fixed point / categorical fixed point

motivation

$$\mathrm{Top} \underset{(-)^H}{\overset{\mathrm{triv}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Top}_G \qquad (-)^H = \mathrm{Map}(G/H_+, -).$$

in spectra

$$Sp \underset{(\,)^H}{\overset{\mathrm{triv}}{\underset{\perp}{\rightleftarrows}}} Sp_G^O. \qquad (\,)^H = F(G/H_+, X).$$

Formal defn: naive fixed points.

$X \in Sp_G^O$ then naive(X) := levelwise G-fixed points of X w/ restricted $O(n)$-action

structure map naive(σ_n) : $S^1 \wedge (X_n)^G \to (X_{n+1})^G$

this naive fixed point doesn't preserve π_*-iso.

Take the right derived functor of this.

$$(\_)^H : Sp_G^O \to Sp$$
$$X \mapsto F(G/H_+, X)$$

bad things about $(-)^H$

- $(X \wedge Y)^H \ncong (X)^H \wedge (Y)^H$
- not commute w/ Σ^∞.

Ex) $H = G$, then $X^G = F_G(S^0, X)$ G-maps.
$= X^G$.

As a special case. X^G capture the G-homotopy type of X.

<u>prop</u> $X \in Sp_G^O$, for every integer k, we have iso
[Schwede, prop 7.2] $\pi_k^G(X) \cong \pi_k(X^G)$.

We still would like to know
$(\Sigma^\infty A)^G = ?$

tom Dieck splitting answers this question.

tom Dieck splitting

Blumberg notes

tells us the categorical fixed points for suspension spectra.

$A \in Top_*^G$

Theorem. $(\Sigma_G^\infty A)^G \cong \bigvee_{(H)\subseteq G} \Sigma^\infty EWH_+ \wedge_{WH} A^H$ (1)

$WH = [G/H, G/H]$ acts by precompose.

$\pi_*^G(\Sigma_G^\infty A) \cong \bigoplus_{(H)\subseteq G} \pi_*^{WH}(\Sigma^\infty EWH_+ \wedge A^H)$ (2)

Adams iso)

index $(H)\subseteq G$ means the sum running over all the conj classes of subgroup $H \subseteq G$.

WH : Weyl group $WH = N_G H/H$ in the case H normal, ~~then~~ $WH = G/H$.

$(EG_+ \wedge X)^G \to X^G \to (\tilde{E}G \wedge X)^G$ is a cofiber seq.

Ex) $G = C_2$

$$(\Sigma_G^\infty A)^{C_2} \cong \Sigma^\infty EC_{2+} \wedge_{C_2} A^e \vee \Sigma^\infty A^{C_2} \cong \Sigma^\infty BC_{2+} \wedge A^e \vee \Phi^{C_2}(\Sigma_{C_2}^\infty A$$

can use (2) to compute $\pi_0^G(S) \cong A(G)$ is iso. of rings / Mackey func

Adams iso.

motivation: $H \overset{i}{\hookrightarrow} G$ induces $i^*: Top_G \to Top_H$

$$\begin{tikzcd} Top_G \arrow[r, "i^*"] & Top_H \arrow[l, bend right, "G_+ \wedge_H -"'] \arrow[l, bend left, "Map^H(G_+, -)"] \end{tikzcd}$$

Adams original statement : $i: H \to G$ $X \in SW^H$ $Y \in SW^G$

(1) $\{X, i^*Y\}_H \longleftrightarrow \{G_+ \wedge_H X, Y\}_G$

(Lewis).

(2) $\{i^*Y, X\}_H \Longleftrightarrow \{Y, G_+ \wedge_H X\}_G$

$$\begin{tikzcd} Sp_G \arrow[r, "i^*"] & Sp_H \arrow[l, bend right, "G_+ \wedge_H -"'] \arrow[l, bend left, "F^H(G_+, -)"] \end{tikzcd}$$

Wirthmüller iso:

$G_+ \wedge_H - \simeq F^H(G_+, -)$.

Adams want to say something about $G \overset{j}{\to} G/N$ $N \triangleleft G$.

$j^*: Sp_{G/N} \to Sp_{G, N\text{-triv.}}$ $X \in SW^G$ $Y \in SW^{G/N}$

X is N-free.

(3) $\{X, j^*Y\}_G \Leftrightarrow \{X_N, Y\}_{G/N}$

alternative notation: X/N $(\_)_N$

X_N = N-orbit space of X.

Adams iso. $\Rightarrow$ (4) $\{j^*Y, X\}_G \Leftrightarrow \{Y, X_N\}_{G/N}$.

Warning: this pair doesn't give adjunction pair of functors as (1) (2)

observation: j^*Y not N-free unless in trivial case.

idea of proof: $(\_)_N$ is functorial for stable G/N-maps.

newest version: [Reich, Varisco]: X good N-free, orthogonal G-spectra

we have the Adams map $E\mathcal{F}(N)_+ \wedge_N X \to X^N$ is a π_*-iso.

$$\mathcal{F}(N) = \{H \leq G \mid H \cap N = \{1\}\}.$$

[LMS86]. generalize this to stable Htpy cat.

If X cofibrant and $E\mathcal{F}(N)_+ \wedge X \to X$ is a w.e. in $Sp_{G, N\text{-triv}}$

then $X_N \simeq (Q^{\mathcal{U}}(X))^N$ $\quad \overbrace{Q^{\mathcal{U}}(X)}^{\Omega\text{-spec}} \xleftarrow{\cong} X$ if X is good.

[RV].

and for every cofibrant $Y \in Sp_{G/N}$.

$$[j^*Y, X]_G \cong [Y, X_N]_{G/N}.$$

sketch of proof). in [MM 02], we have Quillen adjunctions.

$$\begin{tikzcd} Sp_{G,\,N\text{triv}} \arrow[r, bend left, "(-)_N"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[d, shift right, "I_{\text{all}}^{N\text{-triv}}"'] & Sp_{G/N} \arrow[l, "j^*"'] \arrow[l, bend left, "(-)^N"'] \arrow[l, phantom, shift left=3, "\perp"] \\ Sp_G \arrow[u, shift right, "I_{N\text{-triv}}^{\text{all}}"'] & \end{tikzcd}$$

warning: N-free

For spaces, X has a free N action.

$\Updownarrow$

$EF(N)_+ \wedge X \to X$ is a w.e.

So we generalize N-free notion for Spectra as this

$$(EF(N)_+ \wedge X)_N = EF(N)_+ \wedge_N X \quad \text{if the condition is satisfied.}$$

X is good if all structure maps are closed embeddings.

$$X \xrightarrow{\simeq} Q^u(X)$$

Cor of Adams iso. $N = G$. $Y = S$, $X = EG_+$

$$[S, EG_+]_G \leftrightarrow [S, BG_+].$$

$$\pi_*^G(EG) \cong \pi_*(BG).$$

Norm: multiplicative transfer

$H \overset{i}{\hookrightarrow} G$ gives restrictions: $\mathrm{CRingSp}_G \to \mathrm{CRingSp}_H$.

this res functor has a left adjoint.

construction: G fin group. $H \leq G$ and $[G:H] = m$.

$\langle G:H \rangle$ denote the set of m-tuples such that their classes in G/H give a partition of G.

$$\langle G:H \rangle := \{ (g_1, \cdots g_m) \in G^{\times m} \mid G = \bigcup_{i=1}^{m} g_i H \}.$$

the wreath product $\Sigma_m \wr H$ is the semi-direct product $\Sigma_m \ltimes H^m$ w/ mult:

$$(\alpha; h_1, \dots h_m)\cdot(\tau; k_1, \dots k_m) = (\alpha\tau;\ h_{\tau(1)}k_1, \dots h_{\tau(m)}k_m).$$

$\Sigma_m \wr H$ acts from the right on $\langle G:H\rangle$:

$$(g_1, \dots g_m)\cdot(\alpha; h_1, \dots h_m) = (g_{\alpha(1)}h_1, \dots\ g_{\alpha(m)}h_m)$$

$\Sigma_m \wr H$ also acts on $X^{\wedge m}$ (from the left). $X \in Sp_H$.

$$(\alpha; h_1, \dots h_m)\cdot(x_1, \dots x_m) := (h_{\alpha^{-1}(1)}x_1, \dots h_{\alpha^{-1}(m)}x_m).$$

So we get a $\Sigma_m \wr H$-sp $P^m X$.

Defn: The norm of $X \in Sp_H$ is the orthogonal G-spectrum:

$$N_H^G X := \langle G:H\rangle_+ \wedge_{\Sigma_m \wr H} P^m X.$$

Properties of the norm N_H^G:

- Symm mon $\quad N_H^G(X \wedge Y) \cong N_H^G X \wedge N_H^G Y.$

- $K \leq H \leq G$, then
$$N_H^G(N_K^H X) \cong N_K^G X \quad \text{for } X \in Sp_K.$$
- preserves π_*-isos of cofibrant obj in Sp_H.
So it has a left derived functor: $Ho(Sp_H) \to Ho(Sp_G)$.
- w/ geo fixed point. X cofibrant in Sp_H.
$$\Phi^H X \simeq \Phi^G N_H^G X.$$