

Outline

- the category of orthogonal  $G$ -spectra  $\text{Sp}_G^0$ 
  - ↳ tensored & cotensored over  $\text{Top}^G$
  - ↳ suspension spectra. (geo fixed point comm w/  $\Sigma$ ).
  - ↳  $\text{SN}^G$  fully-faithful embedding into  $\text{Sp}_G^0$
  - ↳ equivariant stable homotopy groups &  $\pi_*$ -iso
  - ↳ symm mon (smash prod)
- derived fixed points (categorical fixed points)
  - ↳ not comm w/  $\Sigma$ !
  - ↳ as right adjs to the trivial functor.
- tom Dieck splitting: characterize the categorical fixed points of  $\Sigma^\infty X$ .
- Adams iso.
- Norm: multiplicative transfer.

defn of  $\text{Sp}_G^0$

indexing category  $\mathcal{O}$ : obj fin dim real inner prod spaces.

$$\text{mor: } \mathcal{O}(V, W) = \begin{cases} * & \dim W < \dim V \\ \mathcal{L}(V, W)_+ & \dim W = \dim V \\ \begin{matrix} V \\ \downarrow \\ S^{W-\phi(V)} \end{matrix} & \dim W > \dim V \\ \phi: V \hookrightarrow W \end{cases}$$

Defn: An orthogonal  $G$ -spectrum is a based pts functor

$$\mathcal{O} \rightarrow \text{Top}_*^G$$

we denote the cat of orthogonal  $G$ -spec as  $\text{Sp}_G^0$

unpack: • assign every  $V$  a  $G$  space  $X(V)$

• for pair  $(V, W)$ , have equivariant str map

$$\mathcal{O}(V, W) \wedge X(V) \rightarrow X(W)$$

combining

$$\wedge_{v, w}: S^{W-v} \wedge X(V) \rightarrow X(W) \text{ for every } V \hookrightarrow W$$

$\mathcal{O}(V)$  action on  $X(V)$

integer graded version.  $V \cong \mathbb{R}^n$  for some  $n$ .

Defn: An orthogonal  $G$ -sp  $X$  is determined by.

- a based space  $X_n$  w/ a based  $O(n) \times G$  action for each  $n \geq 0$
- a based  $G$ -eq structure map  $\Lambda_n: X_n \wedge S^1 \rightarrow X_{n+1}$  for each  $n \geq 0$
- for all  $m, n \geq 0$ , the iterated structure map

$$S^m \wedge X_n \xrightarrow{S^{m-1} \wedge \Lambda_n} S^{m-1} \wedge X_{n+1} \rightarrow \dots \rightarrow X_{n+m}$$

is  $O(m) \times O(n)$ -equivariant.

Rmk.). not the same as in HHR.

$\mathcal{O}_G$  obj: fin dim  $G$ -repre

mor:  $\mathcal{O}_G(V, W) = O(V, W)$  plus a  $G$ -action by conj.

Fact: for  $V$  a  $G$ -representation

define  $X(V) := \mathcal{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$

'So can think of  $X$  is representation graded'

$\text{Sp}_G^0$  tensored & cotensored over  $\text{Top}^G$

$A$  a  $G$ -space  $X \in \text{Sp}_G^0$

define  $A \wedge X (V) := A \wedge X(V)$   $G$  acts diagonally.

$\text{Map}(A, X)(V) := \text{Map}(A, X(V))$   $G$  acts by conj.

ex)  $\Sigma^V X := S^V \wedge X$   $\Omega^V X := \text{Map}(S^V, X)$ .

Suspension spectrum

$$\Sigma^\infty A (V) := S^V \wedge A$$

$\text{Sp}_G^0$  fully-faithful embedding into  $\text{Sp}_G^0$

$$[\Sigma^\infty X, \Sigma^\infty Y]_G \cong \text{colim}_{V \hookrightarrow U} [\Sigma^V X, \Sigma^V Y]_G = \{X, Y\}_G.$$

for each  $X, Y$  fin  $G$  CW.

equivariant stable htpy groups of orthogonal  $G$ -spectra.

$H$  closed subgroup of  $G$ .  $X \in \text{Sp}_G^0$   
 $H$ -eq 0-th stable htpy group of  $X$ .

Cary's note

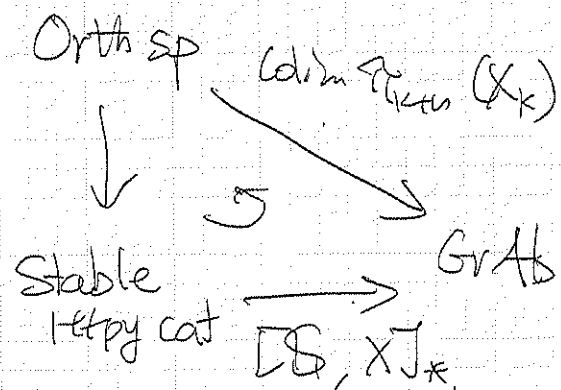
$$\pi_0^H(X) := \text{colim}_{V \in U} [S^V, X(V)]_H$$

non eq

$k$  positive integer.

$$\pi_k^H(X) := \text{colim}_V [S^{V \oplus \mathbb{R}^k}, X(V)]_H$$

$$\pi_{-k}^H(X) := \text{colim}_V [S^V, X(V \oplus \mathbb{R}^k)]_H$$



$\pi_*$ -iso (weak eq). strict model str.

$X \xrightarrow{f} Y \in \text{Sp}_G^0$  is a  $\pi_*$ -iso if

$$\pi_n^H(f) : \pi_n^H(X) \rightarrow \pi_n^H(Y)$$

is iso for all closed  $H \in G$  and all integer  $n$ .

# Smash prod on $\mathbb{S}p_G^0$

Smash prod of non eq + diagonal  $G$ -action.

defn:  $X, Y \in \mathbb{S}p_G^0$   $X \wedge Y \in \mathbb{S}p_G^0$  the  $n$ -th level is the coequalizer

$$\begin{array}{ccc}
 \begin{array}{c} V \\ p+q=n \end{array} \begin{array}{c} O(n)_+ \wedge \\ O(p) \times O(q) \end{array} \wedge (X_p \wedge S^1 \wedge Y_q) & \begin{array}{c} \xrightarrow{\alpha_{p \wedge Y_q}^X} \\ \xrightarrow{\alpha_{p \wedge Y_q}^Y} \end{array} & \begin{array}{c} (V \wedge O(n)_+ \wedge \\ p+q=n \end{array} \wedge (X_p \wedge Y_q) \rightarrow (X \wedge Y)_n \\
 & & \downarrow \cong \\
 & & \mathbb{Z}
 \end{array}$$

$\alpha_{p \wedge Y_q}^X = X_p \wedge \text{bracket}_{S^1, Y_q}$

naturally extends to a functor  $\mathbb{S}p_G^0 \times \mathbb{S}p_G^0 \rightarrow \mathbb{S}p_G^0$

this makes  $(\mathbb{S}p_G^0, \wedge, \mathbb{S})$  a symm mon category.

(closed) has internal hom.

geo fixed point comm w/ suspension

recall  $\phi^G = (\widetilde{E}P \wedge X)^G$

$$\Sigma^\infty A(U) := S^U \wedge A$$

$$\Sigma_G^\infty: G\text{-Top} \rightarrow G\text{-Sp.}$$

$$\phi^G(\Sigma_G^\infty A) \cong \Sigma^\infty(A^G)$$

# derived fixed point / categorical fixed point

motivation

$$\text{Top} \begin{array}{c} \xrightarrow{\text{triv}} \\ \perp \\ \xleftarrow{(-)^H} \end{array} \text{Top}^G$$

$$(-)^H = \text{Map}(G/H_+, -)$$

in spectra

$$\text{Sp} \begin{array}{c} \xrightarrow{\text{triv}} \\ \perp \\ \xleftarrow{(-)^H} \end{array} \text{Sp}^G$$

$$(-)^H = \text{FC}(G/H_+, X)$$

Formal defn: naive fixed points,

$X \in \text{Sp}_G^0$  then  $\text{naive}(X) :=$  levelwise  $G$ -fixed points of  $X$   
w/ restricted  $O(n)$ -action

structure map  $\text{naive}(X_n) = S^1 \wedge (X_n)^G \rightarrow (X_{n+1})^G$

this naive fixed point doesn't preserve  $\pi_*$ -iso.

Take the right derived functor of this.

$$(-)^H: \text{Sp}_G^0 \rightarrow \text{Sp} \\ X \mapsto \text{FC}(G/H_+, X)$$

bad things about  $(-)^H$

- $(X \wedge Y)^H \not\cong (X)^H \wedge (Y)^H$
- not commute w/  $\Sigma^\infty$ .

Ex)  $H = G$ ,  $G$ -maps.  
then  $X^G = F_G(S^0, X)$   
 $\subset X^G$ .

As a special case.  $X^G$  capture the  $G$ -homotopy type of  $X$ .

prop  $X \in \text{Sp}_G^0$ , for every integer  $k$ , we have iso  
[Schwede, Prop 7.2]  $\pi_k^G(X) \cong \pi_k(X^G)$ .

We still would like to know

$$(\Sigma^\infty A)^G = ?$$

tom Dieck splitting answers this question.



# tom Dieck Splitting

Blumberg notes

tells us the categorical fixed points for suspension spectra.

$A \in \text{Top}_*^G$   
Theorem.

$$\begin{aligned}
 (\Sigma_G^\infty A)^G &\cong \bigvee_{(H) \in G} \sum^{\infty} EWH_+ \wedge_{WH} A^H & (1) \\
 \pi_*^G (\Sigma_G^\infty A) &\cong \bigoplus_{(H) \in G} \pi_*^{WH} (\sum^{\infty} EWH_+ \wedge A^H) & (2)
 \end{aligned}$$

$WH = [G/H, G/H]$  acts by precompose.   
 $\left. \begin{array}{l} (1) \\ (2) \end{array} \right\} \text{ Adams iso.}$

index  $(H) \in G$  means the sum running over all the conj classes of subgroup  $H \in G$ .

$WH$ : Weyl group  $WH = N_G H / H$  in the case  $H$  normal,

then  $WH = G/H$ .

$(EG_+ \wedge X)^G \rightarrow X^G \rightarrow (EG \wedge X)^G$  is a cofiber seq.

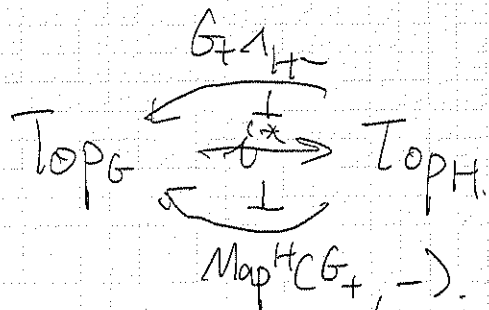
Ex)  $G = C_2$

$$(\Sigma_G^\infty A)^{C_2} \cong \sum^{\infty} EC_{2+} \wedge_{C_2} A^e \vee \sum^{\infty} A^{C_2} \cong \sum^{\infty} BC_2 \wedge A^e \vee \bigoplus^{C_2} (\Sigma_{C_2}^\infty A)$$

can use (2) to compute  $\pi_0^G(S) \cong A(G)$  is iso. of rings / Mackey funic

Adams iso.  
motivation:

$H \xrightarrow{i} G$  induces  $i^*: \text{Top } G \rightarrow \text{Top } H$

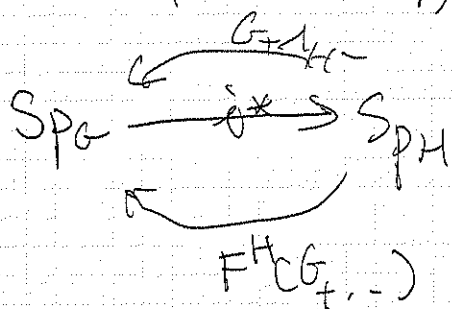


Adams original statement:  $i: H \rightarrow G$   $X \in \text{SW}^H$   $Y \in \text{SW}^G$

(1)  $\{X, i^*Y\}_H \leftrightarrow \{G_+ \wedge_H X, Y\}_G$

(2)  $\{i^*Y, X\}_H \leftrightarrow \{Y, G_+ \wedge_H X\}_G$

(Lewis)



Wirthmüller iso:

$G_+ \wedge_H - = F^H(G_+, -)$

Adams want to say something about  $G \xrightarrow{j} G/N$   $N \triangleleft G$ .

$j^*: \text{Sp}_{G/N} \rightarrow \text{Sp}_{G, N \text{triv.}}$

$X \in \text{SW}^G$   $Y \in \text{SW}^{G/N}$   
 $X$  is  $N$ -free.

$$(3) \{X, j^*Y\}_G \Leftrightarrow \{X_N, Y\}_{G/N}$$

alternative notation:  $X/N \quad (-)_N$

$X_N = N$ -orbit space of  $X$ .

Adams iso.  $\Rightarrow$  (4)  $\{j^*Y, X\}_G \Leftrightarrow \{Y, X_N\}_{G/N}$ .

Warning: this pair doesn't give adjunction pair of functors as (1) (2).

observation:  $j^*Y$  not  $N$ -free unless in trivial case.

idea of proof:  $(-)_N$  is functorial for stable  $G/N$ -maps.

newest version: [Reich, Varisco]:  $X$  good  $N$ -free, orthogonal  $G$ -spectrum

we have the Adams map  $E\mathbb{F}(N)_+ \wedge_N X \rightarrow X^N$  is a  $\pi_X$ -iso.

$$\mathbb{F}(N) = \{H \in G \mid H \cap N = \{1\}\}.$$

[CMS86]. generalize this to stable Hkpy cat.

If  $X$  cofibrant and  $E\mathbb{F}(N)_+ \wedge X \rightarrow X$  is a w.e. in  $Sp_{G, N}$ -triv.

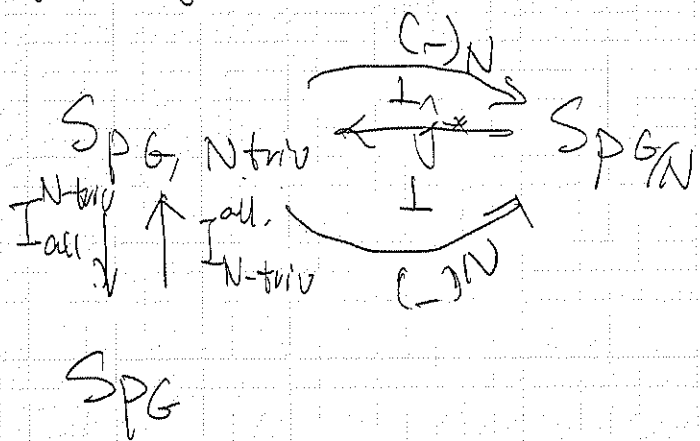
then  $X_N \simeq (Q^u(X))^N \quad Q^u(X) \xleftarrow{\cong} X$  if  $X$  is good.

[RVJ].

and for every cofibrant  $Y \in \mathcal{S}p_{G/N}$ .

$$[j^*Y, X]_G \cong [Y, X_N]_{G/N}$$

sketch of proof). in [MM02], we have Quillen adjunctions.



warning:  $N$ -free

For spaces,  $X$  has a free  $N$  action.

$$(E\mathbb{F}(N)_+ \wedge X \rightarrow X \text{ is a w.e.})$$

So we generalize  $N$ -free notion for Spectra as this

$$(E\mathbb{F}(N)_+ \wedge X)_N = E\mathbb{F}(N)_+ \wedge_N X \text{ if the condition is satisfied.}$$

$X$  is good if all structure maps are closed embeddings.

$$X \xrightarrow{\cong} Q^u(X)$$

Cor of Adams iso.  $N = \mathbb{B}$ .  $Y = \mathbb{S}$ ,  $X = EG_+$

$$[\mathbb{S}, EG_+]_G \leftrightarrow [\mathbb{S}, B\mathbb{G}_+]$$

$$\pi_*^G(EG) \cong \pi_*(BG).$$

Norm: multiplicative transfer

$H \xrightarrow{\hat{i}} G$  gives restrictions:  $\text{CRing Sp}_G \rightarrow \text{CRing Sp}_H$ .

this res functor has a left adjoint.

construction:  $G$  fin group,  $H \leq G$  and  $[G:H] = m$ .

$\langle G:H \rangle$  denote the set of  $m$ -tuples such that their classes in  $G/H$  give a partition of  $G$ .

$$\langle G:H \rangle := \left\{ (g_1, \dots, g_m) \in G^{\times m} \mid G = \bigcup_{i=1}^m g_i H \right\}.$$

The wreath product  $\Sigma_m \wr H$  is the semi-direct product  $\Sigma_m \ltimes H^m$  w/ multi:

$$(a; h_1, \dots, h_m) \cdot (\tau; k_1, \dots, k_m) = (a\tau; h_1 \tau k_1, \dots, h_m \tau k_m)$$

$\Sigma_m \wr H$  acts from the right on  $\langle G: H \rangle$ :

$$(g_1, \dots, g_m) \cdot (a; h_1, \dots, h_m) = (g_1 a h_1, \dots, g_m a h_m)$$

$\Sigma_m \wr H$  also acts on  $X^{\wedge m}$  (from the left),  $X \in \text{Sp}H$ .

$$(a; h_1, \dots, h_m) \cdot (x_1, \dots, x_m) := (h_1 a^{-1} x_1, \dots, h_m a^{-1} x_m)$$

So we get a  $\Sigma_m \wr H$ -sp  $P^m X$ .

Defn: The norm of  $X \in \text{Sp}H$  is the orthogonal  $G$ -spectrum:

$$N_H^G X := \langle G: H \rangle_{+1, \Sigma_m \wr H} P^m X$$

Properties of the norm  $N_H^G$ .

• Symm mon  $N_H^G (X \wedge Y) \cong N_H^G X \wedge N_H^G Y$

•  $K \in H \in G$ , then

$$N_H^G(N_K^H X) \cong N_K^G X \quad \text{for } X \in \mathcal{S}p_K.$$

• preserves  $\pi_*$ -isos of cofibrant obj in  $\mathcal{S}p_H$ .

so it has a left derived functor:  $\text{Ho}(\mathcal{S}p_H) \rightarrow \text{Ho}(\mathcal{S}p_G)$ .

• w/ geo fixed point.  $X$  cofibrant in  $\mathcal{S}p_H$ .

$$\phi_H X \cong \phi_G N_H^G X.$$