Lecture 23 & 24 : Applications

Some references :

Rober Oliver's original paper: A proof of the Conner conjecture
https://www.jstor.org/stable/1970955?seq=1#metadata_info_tab_contents

Weinan Lin's note from 2019 Summer School
https://iwoat.github.io/2019/notes/Lecture-12.pdf

Matthew Scalamandre REU paper
http://math.uchicago.edu/~may/REU2018/REUPapers/Scalamandre.pdf

Goal: " Hope that you can present sth cool in equivariant homotopy theories to your friends who are not here."

Review Smith Theory

Prove Conner Conjecture

Suggestion : Stop me at anytime to ask questions

Review

Slogan : " pts are orbits."

Bredon Cohomology (Talk 13)

A : a coefficient system. i.e $A : \mathrm{Orb}_G^{op} \to \mathrm{Ab}$.

dimension axiom : $\tilde{H}_G^*(G/H_+, A) = \begin{cases} 0 & * \neq 0 \\ A(G/H) & * = 0 \end{cases}$

G/H plays the role of pt.

e.g. $H_G^*(X;\underline{\mathbb{Z}}) \cong H^*(X/G;\mathbb{Z})$ (*)

in particular take $X = EG$, $X/G = BG$.

Exercise 1. Check (*)

Approach 1. Check both sides satisfy the axioms of Bredon cohomologies, then check they agree on 0-dimension. Hence, they are the same.

Approach 2. Compute both sides by defn.

The LHS can be computed via cochain complex.

Recall. for a G-module M, we can define a coefficient system $\underline{M}$ by

$$\begin{tikzcd} \underline{M}(G/K) = M^K \arrow[d] \\ \underline{M}(G/H) = M^H \end{tikzcd} \qquad H \subseteq gKg^{-1} \text{ for some } g \in G$$

In particular, give an abelian grp A we can equip it with the trivial G-action, and $\underline{A}$ gives the constant coeff system

$\underline{\mathbb{Z}}(G/H) = \mathbb{Z}^H = \mathbb{Z}$ in the above eg.

- $$H^*_G(X, A) = H^*(\operatorname{Hom}(\underbrace{\underline{C}_*(X)}_{\text{Coeff sys}}, \underset{\substack{\uparrow \\ \text{Chain of Abel grps}}}{A}))$$

where $\underline{C}_n(X) : \mathrm{Orb}_G^{op} \to Ab$

$G/H \to H_n(X_n^H, X_{n-1}^H; \mathbb{Z})$

Exercise 1' What's the relation btw.

$$H_*^G(X;?) \simeq H_*(X/G;\mathbb{Z}).$$

Why coefficient system?

e.g. in the proof of Smith Theorem (Talk 13)

$G = \mathbb{Z}/p$, X G-space

$X^G \hookrightarrow X \to X/X^G$

cofiber.

$\tilde{H}^*((X/X^G)/G;\mathbb{F}_p)$ By axioms $\tilde{H}^*(X,L)$

$H^*(X;\mathbb{F}_p)$ $\rightsquigarrow$ $H^*(X,M)$

$H^*(X^G;\mathbb{F}_p)$ defined coeff system L, M, N. $H^*(X,N)$

To keep the notation consistent with the references in the beginning

we change the notation M N L in Talk 13 to

A B C

However, to relate A, B, C one does need the coeff system.

to define I : $\text{Orb}_{\mathbb{Z}/p} \to \text{Ab}$

$$\begin{array}{ccc} (\mathbb{Z}/p)/(\mathbb{Z}/p) & & 0 \\ \uparrow & & \downarrow \\ (\mathbb{Z}/p)/e & & I = \ker(\mathbb{F}_p[\mathbb{Z}/p] \xrightarrow{\text{sum of coeff}} \mathbb{F}_p) \end{array}$$

! This is sth we don't have in ordinary cohomology.

There are SES of coefficient system.

$$0 \to I \to A \to B \oplus C \to 0 \qquad (**)$$

$$0 \to C \to A \to B \oplus I \to 0$$

$$(0 \to I^{n+1} \to I^n \to C \to 0)$$

Exercise: Check the above sequences are exact.

e.g. That $(**)$ is exact means

$$\begin{array}{ccccccccc} 0 \to & I(G/G) & \to & A(G/G) & \to & B(G/G) \oplus C(G/G) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & I(G/e) & \to & A(G/e) & \to & B(G/e) \oplus C(G/e) & \to 0 \end{array}$$

Commutes + rows are exact as abelian gps.

the top row $0 \to 0 \to \mathbb{F}_p \xrightarrow{\cong} \mathbb{F}_p \oplus 0 \to 0$

They give LES. Recall in the pf of Smith thm.

we proved

$$\sum_q \dim H^q(X^G) \leq \sum_q \dim H^q(X)$$

(If X has finite dimension.)

Notation
$$a_q := \dim H^q(X)$$
$$b_q := \dim H^q(X^G)$$
$$c_q := \dim \tilde{H}^q((X/X^G)/G)$$
$$i_q := \dim H^q_G(X, I)$$

during the pf we show. (we need later)

$(\star)$ $$2b_q + c_q + i_q \leq 2a_q + c_{q+1} + i_{q+1}$$

Exercise: play with the LES to check it.

<u>Conner Conj.</u>

Oliver Transfer:

If $H \subset K \subset G$

we have $\pi : X/H \to X/K$ (from the proj $G/H \to G/K$)

induces $\pi^*: \tilde{H}^*(X/K) \to \tilde{H}^*(X/H)$

there exists a transfer map

$$\tau : \tilde{H}^*(X/H) \to \tilde{H}^*(X/K)$$

st $\tau\circ\pi^* : \tilde{H}^*(X/K) \to \tilde{H}^*(X/K)$

is multiplication by $\chi(K/H)$

(we have not used any Bredon Cohomology here.)

Recall (Talk 15 & 16)

equivariant: $i : K/H \hookrightarrow V$ (repn.)

$$t : S^V \to Th(\nu) \to Th(\nu \oplus \tau) \simeq K/H_+ \wedge S^V$$

(Tangent of K/H; $\nu\oplus\tau \cong V$)

ν: normal bundle of i

as a tubular neighborhood

sitting in V

$$\chi(K/H) : S^V \xrightarrow{t} K/H_+ \wedge S^V \xrightarrow{proj} S^V$$

the underlying is a degree map

↖ Euler character.

Rk. • G finite. $\chi(K/H) = |K/H|$

• additive transfer (Alg intuition)

Coeff system. ⟶ Mackey functor

$$\begin{array}{ccc} A(G/G) & & A(G/G) \\ \text{res}\downarrow & \rightsquigarrow & \text{res}\downarrow\uparrow\text{tr} \\ A(G/e) & & A(G/e) \end{array}$$

e.g. fixed pt Mackey functor

G-module M

$$\underline{M}(G/H) := M^H.$$

Take $M = \mathbb{Z}[G]$

$$\begin{array}{ccc} M^G & & (\mathbb{Z}[G])^G = \mathbb{Z}\langle \sum_{g\in G} g \rangle \\ \downarrow & \text{res} \downarrow & \uparrow \text{tr} \\ M & & (\mathbb{Z}[G])^e = \mathbb{Z}[G] \end{array}$$

$$\text{tr}: a \to \sum_{g\in G} ga$$

Recall: Thm (Lewis-May-McClure)

C coeff system

$H_G^*(-, C)$ Bredon cohomology $* \in \mathbb{Z}$

(repn in a universe)

Then $H_G^*(-, C)$ can extend to an $RO(G)$-graded one

iff C can extend a Mackey functor.

Fact. A abelian grp.

Constant coefficient system $\underline{A}$

can extend to a Mackey functor $\underline{A}$

Conner Conjecture (proved by Oliver)

If G cpt lie grp, A abelian grp.

X G-CW cplx

① finite dim

② finitely many type of orbits

then $\tilde{H}(X;A)=0 \Rightarrow \tilde{H}(X/G;A)=0$

Rk. X finite dim is necessary. Counter ex $X=EG$, $X/G=BG$. $\tilde{H}^*(EG,A)=0$

$G=\mathbb{Z}/2$ $BG=\mathbb{RP}^\infty$ $H^*(BG,\mathbb{F}_2)=\mathbb{F}_2[x]$

pf ingredients
- Smith Thm. (techniques in the pf) Bredon Cohomology
- Transfer map

Idea: Build up from easier cases.

Case 1. $G=\mathbb{Z}/p$, $A=\mathbb{F}_p$

$X^G \to X \to X/X^G$

$\mathbin{\rotatebox{90}{$\circlearrowleft$}}$ G orbit

$X^G \to X/G \to (X/X^G)/G$

WTS X/G acyclic

It's enough to show X^G, $(X/X^G)/G$ are acyclic.

Now we prove that X^G is acyclic.

X^G follows from $\sum \dim \tilde{H}^q(X^G) \le \sum \dim \tilde{H}^q(X)=0$ $\quad$ X acyclic

$\Rightarrow X^G$ acyclic.

$(X/X^G)/G$: correspond to coeff system C.

Recall that we have,

$$c_q + i_f + 2\sum_{i=f}^{r} b_i \le 2\sum_{i=f}^{r} a_i$$

Ex. Recall the above from the pf in Smith Theory.

note that: b_i, a_i

$X^{\prime G}$ $X^\prime$ are both acyclic.

$\Rightarrow$ $\sum b_i = \sum a_i = 0$

$\Rightarrow$ $c_f + i^* c_f \leq 0$

$c_f = 0$

$\Rightarrow$ $(X/X^G)/G$ acyclic.

Finish the case $G = \mathbb{Z}/p$ $A = \mathbb{F}_p$

Case 2. G finite p-grp $A = \mathbb{F}_p$

$\Downarrow$

G solvable.

$1 \lhd G_1 \lhd G_2 \cdots \lhd G$

G_{i+1}/G_i abelian.

can assume $G_{i+1}/G_i = \mathbb{Z}/p$

induction reduces to case 1

$\Rightarrow$ G finite p grp $A = \mathbb{F}_p$ case ✓

Case 3 G finite $A = \mathbb{F}_p$

Need transfer!

denote a p-Sylow grp of G as P

transfer. $X/P \to X/G$

$$\underbrace{H^*(X/G, \mathbb{F}_p) \xrightarrow{\pi^*} H^*(X/P, \mathbb{F}_p) \xrightarrow{\Sigma} H^*(X/G, \mathbb{F}_p)}_{\text{iso}}$$

$$\Sigma \circ \pi^* = - \times \chi(G/P)$$

$$= - \times |G/P|$$

$\nearrow$ positive integer coprime to p

$\Rightarrow$ iso in $\mathbb{F}_p$ coefficients

$H^*(X/P, \mathbb{F}_p) \cong$ by case 2

$\Rightarrow H^*(X/Q, \mathbb{F}_p) \cong$.

Case 4. G finite ~~$A=\mathbb{Q}$~~

Similar to Case 3.

$$H^*(X/G, \mathbb{Q}) \xrightarrow{\pi^*} H^*(X/e, \mathbb{Q}) \xrightarrow{\tau} H^*(X/G, \mathbb{Q})$$

$\times |G|$ iso in $\mathbb{Q}$

the middle term is 0 by the condition.

$\Rightarrow H^*(X/G, \mathbb{Q}) \cong 0$

Case 3+4 $\mathbb{F}_p$, $\mathbb{Q}$ acyclic $\Rightarrow$ G finite

universal coeff Thm, A all abelian grps:

$\tilde{H}^*(X, A) \cong 0 \Rightarrow \tilde{H}^*(X/G, A) \cong 0$.

for all $*$ for all $*$.

Case 5 $G = S^1$ $\mathbb{F}_p$ coefficient.

(finitely many orbits)

The condition there finitely many type of orbits means $\{G/H\}$ appear in X, there are only finitely many H there.

$G = S^1$, Hence there $\exists$ finite cyclic $C \subset S^1$ st $X^C = X^{S^1}$

for C we can use the Case 3+4

$\Rightarrow X^C$, X/C are acyclic. ($\mathbb{F}_p$)

$\Rightarrow \quad X^{S^1} = X^{C}$ is $\mathbb{F}_p$-acyclic.

use the same trick in case 1

$\Rightarrow \quad (X/X^{S^1})/S^1$ is $\mathbb{F}_p$-acyclic.

$\Rightarrow \quad X^{S^1}$ is $\mathbb{F}_p$-acyclic.

Case 6 $\quad G \cong S^1 \quad \mathbb{Q}$ coefficient.

Apply a rational version smith theorem,

we can prove this case similarly as case 5.

ex. Check this case.

Case 5+6 $\quad G = S^1 \quad$ for all abelian grp. coefficient.

Case 7. $\quad G = T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \cdot \text{copy}}$

$S^1 = T \triangleleft \cdots \; T^{n-1} \triangleleft T^n$

induction reduce to case 6+5.

Case 8. $\quad G$ cpt lie

Fact: G connected cpt lie grp. with maximal torus T^n

Then. $\chi(G/N_G(T^n)) = 1$

Consider.

$$H^*(X/N_G(T^n)) \xrightarrow{\pi^*} H^*(X/G) \xrightarrow{t} H^*(X/N_G(T^n))$$

$\qquad$ iso.

But X acyclic $\Rightarrow X/T^n$ acyclic (case 7)

$\Rightarrow X/N_G(T^n)$ acyclic (case 3+4)

+ the fact $N_G(T^n)/T^n$ is a finite. in a cpt lie grp G with maximal torus T^n.

ex. check that $N_G(T^n)/T^n$ is finite in cpt Lie grp G.
(T^n is maximal torus in G)

$H^*(X/N_G(T)) = 0 \Rightarrow H^*(X/G) = 0.$

finish the G cpt Lie grp case.