Equivariant K-theory

Sunday, July 25, 2021 2:35 PM

Contents:

} Part 1.

Part 2.

Notes:

- My notes from the summer school in 2019 (only the first part)
- Agnes Beaudry's lecture in the 2016 Talbot workshop

References:

- Greenlees, Chapter XIV in *Equivariant Homotopy and Cohomology Theory* (aka the Alaska notes)
- Atiyah, *K-theory and reality*
- Dugger, *An Atiyah-Hirzebruch spectral sequence for KR-theory*
- Heard and Stojanoska, *K-theory, reality, and duality*

1. Non-equivariant K-theory.

X = compact manifold.

$KO(X) = (\text{real vector bundle over } X, \oplus)^{+}$ ← gp completion

$K(X) = KU(X) = (\text{complex vector bundle over } X, \oplus)^{+}$

Facts: $KO(X)$ and $K(X)$ are rings. w/ multiplication induced by $\otimes$.

- $KO(pt) = \mathbb{Z}$ $\quad$ $\mathbb{R}^n$ or $\mathbb{C}^n$ $\downarrow$ pt $\longleftrightarrow n \in \mathbb{Z}$.
  $KU(pt) = \mathbb{Z}$.

$\widetilde{KU}(X) = \operatorname{coker}(K(pt) \xrightarrow{p^*} K(X))$.

$= \{$stable isom classes of complex vector bundles over $X\}$

i.e. $[E_1] = [E_2]$ in $\widetilde{KU}(X)$ if

$E_1 \oplus \mathbb{R}^{n_1} = E_2 \oplus \mathbb{R}^{n_2}$ for some n_1, n_2

☆ Bott periodicity.

$\left[\begin{array}{c} \mathcal{O}(-1) \\ \downarrow \\ S^2 = \mathbb{CP}^1 \end{array}\right] = H$. $\beta := [H] - 1 \in \widetilde{K}(S^2)$.

- $\widetilde{K}(S^2) = \mathbb{Z}\{\beta\}$.

$S^2 = \mathbb{C}P^1$

- $\tilde{K}(S^2) = \mathbb{Z}\{\beta\}$.
- $\tilde{K}(X) \xrightarrow[\sim]{\beta \boxtimes (-)} \tilde{K}(S^2 \wedge X)$

$\Rightarrow$ Can extend $\tilde{K}$ to a reduced gen cohomolgy thy:

$$\tilde{K}^{-i}(X) := \operatorname*{colim}_{n\to\infty} \tilde{K}(S^{2n} \wedge S^i \wedge X)$$

$\tilde{K}$ is represented by.

$$KU(X) = [X, BU \times \mathbb{Z}].$$

Bott periodicity: $\cdot\ \Omega BU = U$.

$\cdot\ \Omega U = BU \times \mathbb{Z}$.

$$\Rightarrow \pi_* KU = \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

$i \bmod 8$	0	1	2	3	4	5	6	7	8
$\pi_i KO$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

In part 4 of the talk we'll relate $\pi_* KU$ & $\pi_* KO$ via homotopy fixed-point spectral sequence.

Once we know $\pi_* KU$ & $\pi_* KO$, we can compute $KU^*(X)$ $KO^*(X)$ by Atiyah–Hirzebruch spectral sequence.

$$E_2^{p,q} = H^p(X; k^q(*)) \Rightarrow K^{p+q}(X)$$

2. Classical equivariant K-theory.

G = Compact Lie group.

X = G-manifold = manifold w/ G-action.

Defn: A G-vector bundle is a map $\xi: E \to X$ s.t.

- $\xi: E \to X$ is a complex vector bundle.
- E is a G-space, ξ is a G-map.
- $g \in G$: $g: E_x \to E_{gx}$ complex linear

· $g \in G$: $g: E_x \to E_{gx}$ complex linear

$K_G(X) = \left(\begin{matrix} G\text{-vector bundles} \\ \text{over } X \end{matrix}, \ \oplus \ \right)^{+}$ ← gp completion

Properties: · $K_G(X)$ is a ring w/ multiplication induced by $\otimes$.

· $K_G(pt) = R(G)$ cpx rep ring of G.
$KO_G(pt) = RO(G)$ real rep ring of G.

Defn: A G-bundle is trivial if it is trivial non-equivariant.

Peter-Weyl's thm ⟹ Any G bundle on X is a summand of a trivial one.

$\tilde{K}_G(X) := \operatorname{coker}(K_G(pt) \xrightarrow{p^*} K_G(X))$.

= Stable isom classes of G-bundles over X.

$[E_1] = [E_2] \in \tilde{K}_G(X)$ iff
$E_1 \oplus V_1 = E_2 \oplus V_2$ for some $V_1, V_2 \in R(G)$

Prop: · If X is G-free, then
$K_G(X) = K(X/G)$.

· If $X = X^G$ (trivial action), then
$K_G(X) = K(X) \otimes_{\mathbb{Z}} R(G)$.

$i: H \hookrightarrow G$. $i^*: K_G(X) \to K_H(X)$ restriction.
$K_H(X) = K_G(G/H \times X)$.

G acts diagonally.
$G/H \times X \to X$ G-equivariant.

When $|G/H| < \infty$, we have a transfer map.
$\operatorname{Hom}_H(G, -): K_H(X) \to K_G(X)$.

When $|G/H|$ infinite, need tools to define.

When $|G/H|$ infinite, need tools to define transfers
(holomorphic / smooth induction).

For non compact space X. $K_G(X) = \tilde{K}_G(X_+)$

Thm: Let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a G-vector bundle over a space X.

Then we have a Thom-isomorphism:

$$\phi : K_G(X) \xrightarrow{\sim} K_G(E).$$

$X = pt$, $E = V$ G-rep

$\lambda(V) = 1 - V + \Lambda^2 V - \cdots + (-1)^{\dim V} \Lambda^{\dim V} V.$

$\in R(G) = K_G(pt).$

$e_V : S^0 \to S^V$ Euler class

$b_V = \phi(1) \in \tilde{K}_G(S^V).$

Thm: $\phi : \tilde{K}_G(X_+) \xrightarrow{\sim} \tilde{K}_G(\Sigma^V X_+).$

$1 \longmapsto b_V.$

$e^V : S^0 \to S^V.$

$e^V \wedge 1 : X_+ \to \Sigma^V X_+.$ $\underline{(e^V)^* b_V = \lambda(V).}$

Strategy:

- Line bundle: clutching function.
- G = ab gp. $R(G)$ is generated by linear reps of G.
  $\rightsquigarrow$ can reduced to the line bundle case.
- $U(n)$: need to use holomorphic transfer
  - max torus & weyl gp.
- G: change of gp.

How about KO_G? Bott periodicity holds

· G: Change of gp.

How about KO_G? Bott periodicity holds for spin reps whose dim is divisable by 8.

Consequence: Can extend $K_G(-)$ to a cohomology theory.

$$K^0_G(X_+) = K_G(X) \quad \text{for fin } G\text{-CW}.$$

$V = \mathbb{C}$ trivial rep. $K^{2n}_G(X) = K^0_G(X)$

$$K^{2n+1}_G(X) = K^0_G(\Sigma X).$$

[Can be extended to a $RO(G)$-theory].

Atiyah - Segal completion thm.

· $K_G(pt) = R(G)$.

I_G = augmentation ideal.

$$= \ker\left(R(G) \xrightarrow{\dim} \mathbb{Z} \right).$$

X = based G-space. $\pi: EG_+ \wedge X \to X$

Thm. $K^*_G(X) \xrightarrow{\pi^*} K^*_G(EG_+ \wedge X)$ is the completion map at I_G.

In particular, $X = pt$.

$$K_G(pt) = R(G) \longrightarrow K_G(EG_+) = K(EG_+/G) = K(BG).$$

Atiyah-Segal completion: $K(BG) = R(G)^\wedge_I$.

Recall,

Atiyah-Segal completion: $K(BG) = R(G)^{\wedge}_{I}$.

Recall homotopy fixed point:

$$X^{hG} = \mathrm{map}(EG_+, X)^G.$$

Borel equivariant K-theory:

$K_G(EG_+ \wedge X)$. $\leftarrow$ htpy fixed pt.

$K_G(X)$ $\leftarrow$ fixed pt.

3. Atiyah's $K\mathbb{R}$ theory. $G = \langle 1, g \rangle$.

Defn: A $\mathbb{R}$eal bundle over a C_2-space X is a map: $\xi: E \to X$ s.t.

- $\xi: E \to X$ is a complex vector bundle.
- ξ is C_2-equivariant.
- $g: E_x \to E_{gx}$. complex <u>anti</u>-linear.

[Comparison: For a C_2-bundle: $g: E_x \to E_{gx}$ is complex linear]

$$K\mathbb{R} := (\mathbb{R}\text{eal vector bundle over } X, \oplus)^+.$$

Why $K\mathbb{R}$? It sees both KU and KO.

Prop: If C_2 acts trivially on X, then $K\mathbb{R}(X) = KO(X)$.

Pf:

$$\begin{array}{ccc} \{\mathbb{R}\text{eal v.b. over } X\} & & \{\text{real v.b over } X\} \\ \begin{array}{c} E \\ \downarrow \\ X \end{array} & \longmapsto & \begin{array}{c} E^{C_2} \\ \downarrow \\ X \end{array} \end{array}$$

$E \otimes \mathbb{C}$ $\circlearrowleft$ C_2 acts by conjugation. $\longleftarrow$ E real

$$\begin{array}{ccc} E \otimes_{\mathbb{R}} \mathbb{C} \circlearrowleft C_2 \text{ acts by conjugation.} & \longleftarrow & E \\ \downarrow & & \downarrow \\ X & & X \end{array}$$

real bundle.

Equivalence of categories. Compatible w/ direct sum.

Prop: $KR(X \times C_2) \underset{\{\pm 1\}}{=} KU(X)$.

Pf: $\{\mathbb{R}\text{- v.b over } X \times C_2\}$ $\quad\quad$ $\{\text{Complex v.b. over } X\}$

$$\begin{array}{ccccc} i_1^* E & & i_{-1}^* E & \longmapsto & i_1^* E \\ \downarrow & & \downarrow & & \downarrow \\ X \times \{1\} & \sqcup & X \times \{-1\} & & X \end{array}$$

$$\begin{array}{ccccc} E & \sqcup & \overline{E} & \longleftarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ X \times \{1\} & \sqcup & X \times \{-1\} & & X \end{array}$$

Equivalence of symm monoidal cat. $\square$.

Bott - periodicity.

$\rho =$ reg representation of $C_2 = \mathbb{C} \circlearrowleft$ complex conjugation

$\widetilde{KR}(X) \simeq \widetilde{KR}(S^\rho \wedge X)$.

$1 \longmapsto \beta \boxtimes 1$.

$H = \left[\begin{array}{c} O(1) \\ \downarrow \\ S^\rho \end{array}\right] = \mathbb{CP}^1 \circlearrowleft$ cpx conjugation. $\quad\quad \beta = [H] - 1$.

$$KR^{s+t\sigma}(X) = \operatorname*{colim}_{n \to \infty} KR(S^{n\rho} \wedge S^{-s-t\sigma} \wedge X).$$

KR is $RO(C_2)$ graded cohomology theory.

$RO(C_2) = \mathbb{Z}\langle 1, \sigma \rangle \quad\quad \sigma =$ sign rep.

$\rho = 1 + \sigma$.

KR is represented by a genuine C_2-
Ω-spectrum.
$\circlearrowleft C_2$. cpx conjugation.

$BU(n) = Gr_n(\mathbb{C}^\infty)$.

$\mathrel{\rotatebox{90}{$\circlearrowleft$}}$

$BU = \operatorname{colim}_{n\to\infty} BU(n)$.

non-equivariantly $\quad \Omega^2 BU = BU \times \mathbb{Z}$.

C_2-equivariantly $\quad \Omega^\rho B_{C_2}U = B_{C_2}U \times \mathbb{Z}$.

Another periodicity:

$KR(S^8) = KO(S^8) = \mathbb{Z}$.
$\rightsquigarrow$ KR has 8-periodicity.

Aside: Very similar to Clifford algebra:

$$Cl_n \longleftrightarrow KO$$
$$\mathbb{C}l_n \longleftrightarrow KU.$$
$$Cl_{s,t} \longleftrightarrow KR.$$

4. KR-computations. $\quad \star \in RO(C_2)$.

Goal: compute $\underline{\pi}^{C_2}_{\star} KR$ as a Mackey functor.

Recall: $V = C_2$-representation.
$B =$ finite C_2-set.

$$\underline{\pi}^{C_2}_V KR(B) = [S^V \wedge B_+, KR]_{C_2}.$$

. Determined by $\underline{\pi}^{C_2}_{*} KR$, $* \in \mathbb{Z}$.

Why? Bott-periodicity.

$$\pi^{C_2}_{[illegible]} KR = \pi^{C_2}_{[illegible]} KR.$$

$$\underline{\pi}^{C_2}_{a+b\sigma} KR = \underline{\pi}^{C_2}_{a-b+b+b\sigma} KR$$

$$= \underline{\pi}^{C_2}_{a-b+b\rho} KR$$

$$= \underline{\pi}^{C_2}_{a-b} KR \qquad a-b \in \mathbb{Z}.$$

$$\underline{\pi}_* KR(C_2/e) = \pi^e_* KR = \pi_* KU.$$

$$\underline{\pi}_* KR(C_2/C_2) = \pi^{C_2}_* KR = \pi_* KR^{C_2}.$$

$$\pi^{C_2}_* KR = \pi_* KO.$$

$$\rightsquigarrow \quad KR^{C_2} = KO.$$

Thm (Homotopy fixed point theorem).

$$KR^{C_2} = KR^{hC_2}.$$

Pf: Recall the isotropy separation sequence.

G = finite gp. $\mathcal{P}$ = Class of proper subgps

$$E\mathcal{P}_+ \to S^0 \to \tilde{E}\mathcal{P}.$$

$$E\mathcal{P}_+^H = \begin{cases} * & H \subsetneq G. \\ \phi & H = G. \end{cases}$$

$$\tilde{E}\mathcal{P}^H = \begin{cases} * & H \subsetneq G. \\ S^0 & H = G. \end{cases}$$

$G = C_2$. $\quad E\mathcal{P} = EC_2.$

$$EC_{2+} \to S^0 \to \tilde{E}C_2 = S^{\infty\sigma}.$$

$$EC_{2+} = S(\infty\sigma)_+$$

Unit sphere in $\mathbb{R}^{\infty\sigma}$

Recall:

$$\begin{array}{ccccc} X_{hG} & \to & X^G & \to & \Phi^G X \\ \downarrow \wr & & \downarrow & & \downarrow \\ X_{hG} & \xrightarrow{\text{Norm}} & X^{hG} & \to & X^{tG} \end{array}$$

want to show $X = K\mathbb{R}$ $G = C_2$. $X^G \xrightarrow{\sim} X^{hG}$.

suffices to show $\Phi^G X = *$

If so, X^{tG} is a module over $*$.

$\Rightarrow X^{tG} = *$

$\Rightarrow X^G \xrightarrow{\sim} X^{hG}$ wk Eq.

$\Phi^G X = (\tilde{E}\mathcal{P} \wedge X)^G$.

Claim: $\tilde{E}C_2 \wedge K\mathbb{R} \simeq *$.

$\tilde{E}C_2 = S^{\infty\sigma} = \underset{n\to\infty}{\mathrm{colim}}\, S^{n\sigma}$

$= \mathrm{colim}\,(S^0 \xrightarrow[e_\sigma]{} S^\sigma \xrightarrow{1\wedge e_\sigma} S^{2\sigma} \to S^{3\sigma} \cdots)$

$= S^0[e_\sigma^{-1}]$

$$K\mathbb{R} \wedge S^0 \xrightarrow{e_\sigma} K\mathbb{R} \wedge S^\sigma$$

$1 \wedge \Sigma^{-1}\eta^{top}$; $\uparrow \beta$ $\rightsquigarrow$ Bott element invertible.

$$K\mathbb{R} \wedge S^{-1}$$

$\eta^{top} \in \pi_1(S^0)$ is nilpotent.

$\Rightarrow S^0[e_\sigma^{-1}] \wedge K\mathbb{R} \simeq *$.

$\rightsquigarrow$

$$K\mathbb{R}_{hC_2} \xrightarrow{\sim} K\mathbb{R}^{C_2} \to \Phi^{C_2} K\mathbb{R} = *$$

$$\begin{tikzcd}
KR_{hC_2} \arrow[r, "\sim"] \arrow[d, "s"] & KR^{C_2} \arrow[r] \arrow[d, "s"] & \Phi^{C_2} KR \arrow[d] \\
KR_{hC_2} \arrow[r, "\sim", "\mathrm{Norm}"'] & KR^{hC_2} \arrow[r] & KR^{tC_2} = *
\end{tikzcd}$$

For KR: Norm: $KR_{hC_2} \xrightarrow{\sim} KR^{hC_2}$.

" Ambidexterity "

$KO \simeq KR^{C_2} \simeq KR^{hC_2}$.

$\Rightarrow$ Homotopy fixed point spectral sequence.

Step up: G = (pro-) finite gp acting on a spectrum X.

Then there is a spectral sequence.

$$E_2^{s,t} = H^s(G; \pi_t(X)) \Rightarrow \pi_{t-s}(X^{hG}).$$

Idea: $X^{hG} = \mathrm{Map}(EG_+, X)^G$

$$EG_\cdot = \left| G \rightleftarrows G\times G \Rrightarrow G\times G\times G \cdots \right|$$

$\qquad 0 \qquad 1$

$\mathrm{Map}(G^{\cdot+1}, X)^G$ cosimplicial set.

Recall $H^*(G; M)$.

$M \to \mathrm{Map}(G, M) \to \mathrm{Map}(G\times G, M) \to \mathrm{Map}(G\times G\times G, M) \to$

HFPSS for $KO = KR^{hC_2} = KU^{hC_2}$.

$$E_2^{s,t} = H^s(C_2; \pi_{2t}(KU)) \Rightarrow \pi_{2t-s}(KO)$$

Q: How does C_2 act on KU.

A: $\pm 1 \in C_2$ acts by the Adams operation $\psi^{\pm 1}$.

$(KU)_* = \mathbb{Z}[v^{\pm 1}]$.

$\psi^1 = id$. $\psi^{-1}(v) = -v$.

$\pi_{2t}(KU) = \mathbb{Z}\{v^t\} \circlearrowleft \psi^{-1}$ by $(-1)^t$.

$\Rightarrow \pi_{2t}(KU) = \begin{cases} \mathbb{Z} & \text{trivial action} \quad t \text{ even} \\ \mathbb{Z}_- & \text{sign rep} \quad t \text{ odd} \end{cases}$

Now compute: $H^s(C_2; \mathbb{Z})$. $H^s(C_2; \mathbb{Z}_-)$.

$$H^s(C_2;\mathbb{Z}) = H^s(BC_2; \mathbb{Z}) = H^s(\mathbb{RP}^\infty; \mathbb{Z}). \qquad H^s(C_2;\mathbb{Z}_-) = H^s(\mathbb{RP}^\infty; \{\mathbb{Z}\})$$

local coeff.

$= \mathbb{Z}[x]/2x$. $|x| = 2$

$E_2^{s,2t} = H^s(C_2; \mathbb{Z}\{v^t\}). = \begin{cases} \mathbb{Z} & s=0, \ t \text{ even} \\ \mathbb{Z}/2 & t-s \text{ even}. \ s>0. \\ 0 & \text{else}. \end{cases}$

Q: Why that d_3-differential?

A: Method 1: "Cheating": we know $\pi_*(KO)$.

Method 2: Fact: KO is a ring spectrum.

Unit: $S^0 \to KO$.

Hurewicz map. $h: \pi_*(S^0) \to \pi_*(KO)$.

η^{top}.

$(\eta^{top})^4 = 0. \rightsquigarrow h(\eta^{top})^4 = 0.$

$(\eta^{top})^4 = 0 \rightsquigarrow h(\eta^{top})^4 = 0$

Method 3: Compare HFPSS.

$\underset{\sim}{(S^{2\rho})^{hC_2}}$ $\quad$ $\underset{\sim}{KR^{hC_2}}$

Fact: The HFPSS for $(\Sigma^{2-2\sigma} K\mathbb{R})^{hC_2} \simeq \Sigma^4 KO$ has the same E_2-page as that for $K\mathbb{R}^{hC_2}$, but the d_3-differential is different!

Slice S.S. to compute $\underline{\pi}_\bigstar^{C_2} K\mathbb{R}$.

$$\underline{E}_2^{s,2t} = \underline{\pi}_{2t-s} P_{2t}^{2t} K\mathbb{R} \Rightarrow \underline{\pi}_{2t-s} K\mathbb{R}$$

$$\| $$

$$\begin{cases} H^{C_2}_{t-s}(S^{t\sigma}; \underline{\mathbb{Z}}) & t \geq 0 \\ H^{s-t}_{C_2}(S^{-t\sigma}; \underline{\mathbb{Z}}) & t < 0 \end{cases}$$

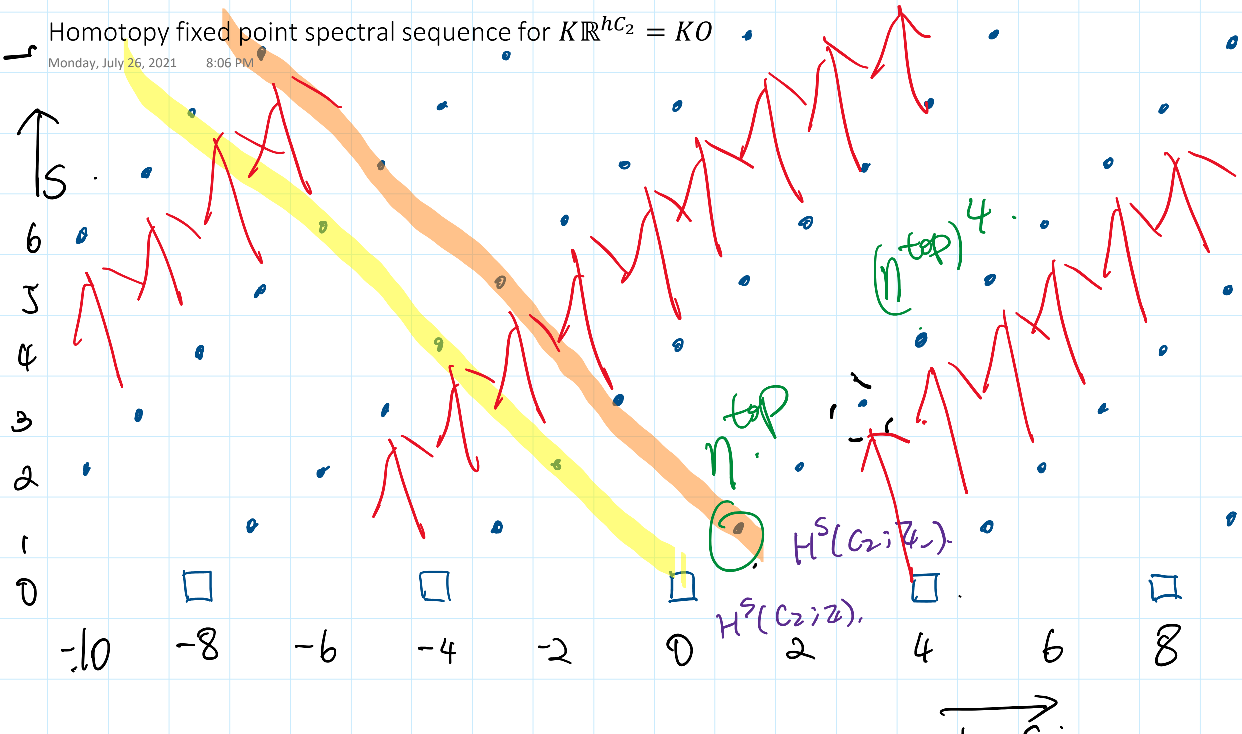
$$S(t\sigma)_+ \to D(t\sigma)_+ \to S^{t\sigma} \qquad /C_2$$

$$\rightsquigarrow \mathbb{RP}^{t-1}_+ \to S^0 \to S^{t\sigma}/C_2$$

essentially $\mathbb{RP}^{t-1}$ (co)homology.

Homotopy fixed point spectral sequence for $K\mathbb{R}^{hC_2} = KO$

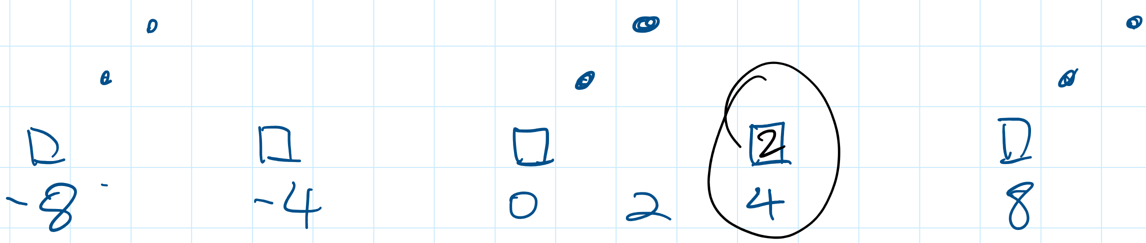
Monday, July 26, 2021 8:06 PM



□ = $\mathbb{Z}$ • = $\mathbb{Z}/2$ Adams grading.

Claim: There is a d_3-differential.

E_4-page collapses. - no extension problem.



$\pi_* KO$: $\mathbb{Z}/2\ \mathbb{Z}/2\ 0\ \mathbb{Z}\ 0\ 0\ 0\ \mathbb{Z}\ \mathbb{Z}/2\ \mathbb{Z}/2\ 0\ \mathbb{Z}$ - - - ...

$$\pi_* KO = \mathbb{Z}[\eta, w, b^{\pm 1}] / [2\eta, \eta^3, w^2 = 4b].$$

$$|\eta| = 1,\ |w| = 4,\ |b| = 8.$$

Slice spectral sequence for $K\mathbb{R}$

Monday, July 26, 2021 8:06 PM