Plan: ① non-equivariant cobordism

② . Real cobordism

③ . Equivariant cobordism.

Thom spectrum, Pontrjagin-Thom construction, Thom Theorem.

M: closed manifold

stable normal bundle of M. (we want some structures).

Def. A (B,f)-structure.

① for $n \geq 0$, $B_n \in Top_*^{CW}$.

② . Serre fibration f_n: $\begin{array}{c} B_n \\ \downarrow f_n \\ BO(n) \end{array}$.

③ . $n_1 \leq n_2$, $\exists$
$$\begin{array}{ccc} B_{n_1} & \xrightarrow{cts} & B_{n_2} \\ f_{n_1} \downarrow & & \downarrow f_{n_2} \\ BO(n_1) & \hookrightarrow & BO(n_2) \end{array}$$

A (B,f)-structure is multiplicative, if

μ:
$$\begin{array}{ccc} B_{n_1} \times B_{n_2} & \longrightarrow & B_{n_1+n_2} \\ f_{n_1} \times f_{n_2} \downarrow & & \downarrow f_{n_1+n_2} \\ BO(n_1) \times BO(n_2) & \longrightarrow & BO(n_1+n_2) \end{array}$$

$\{\mu\}$: associative + unital. $B_0 = *$.

Def: Let M is a n-manifold, a (B,f)-structure on the stable normal bundle of M, is an equivalent class (i, g):

$$i: M \hookrightarrow \mathbb{R}^k$$

g is a htpy class of lift

$$\begin{array}{ccc} & & B_{k-n} \\ & \overset{g}{\nearrow} & \downarrow f_{k-n} \\ X & \longrightarrow & BO(k-n) \end{array}$$

classifying map of normal bundle of $(X \overset{i}{\hookrightarrow} \mathbb{R}^k)$

equivalence generated by

$$\begin{tikzcd}
M \arrow[r, hook, "i_1"] & \mathbb{R}^{k_1} \arrow[r, hook] & \mathbb{R}^{k_2}.
\end{tikzcd}$$

$$\begin{tikzcd}
 & B_{k_1-n} \arrow[r] \arrow[d] & B_{k_2-n} \arrow[d] \\
M \arrow[ur, "g_1"] \arrow[urr, "g_2"'] \arrow[r] & BO(k_1-n) \arrow[r] & BO(k_2-n)
\end{tikzcd}$$

Ex/. ① unoriented manifold

$B_n = BO(n) \xrightarrow{id} BO(n)$ MO

② oriented manifold

$B_n = EO(n)/SO(n) \to BO(n)$ MSO

③ framed manifold

$B_n = EO(n) \to BO(n)$ S

④. $B_{2n} = BU(n) \to BO(2n)$. MU

this is a (B,f)-structure for $BO(\text{even})$. $S^{\mathbb{C}} \wedge MU(n) \to MU(n+1)$.

Def. Two closed manifold M_1, M_2 of dim n, w/ (B,f)-structure on stable normal bundle are bordant, if there exits $(n+1)$-dimension W, w/. (B,f)-str on ~~normal~~ stable normal bundle, s.t. $\partial W \simeq M_1 \sqcup (-M_2)$.

← some onstruction

Bordant relation is an equivalent relation. Equivalent classes form a graded commutative ring Ω_*^B.

$+$: disjoint union

$\times$: Cartesian product.

We will see $\Omega_*^B \cong \pi_* MB$.

↑ Thom spectrum associated to (B,f)-structure.

Given a vector bundle ξ, Thom space of ξ, $Th(\xi) := D(\xi)/S(\xi)$.

Prop. $Th(\xi_1 \oplus \mathbb{R}^n) = \Sigma^n Th(\xi)$.

$\xi_1 : E_1 \to B_1$

$\xi_2 : E_2 \to B_2$

$B_1 \times B_2 \xrightarrow{p_1} B_1$, $B_1 \times B_2 \xrightarrow{p_2} B_2$

$Th(p_1^* \xi_1 \oplus p_2^* \xi_2) = Th(\xi_1) \wedge Th(\xi_2)$.

Thom spectrum.

Given a (B, f) structure, there is a universal n-bundle. pullback of universal bundle over $BO(n)$ along $B_n \xrightarrow{f_n} BO(n)$.

$(MB)_n := Th(f_n^* \gamma^n)$ γ^n: universal n-bundle over $BO(n)$.

Prop. pullback along

$$\begin{array}{ccc} B_n & \longrightarrow & B_{n+1} \\ \downarrow & & \downarrow \\ BO(n) & \longrightarrow & BO(n+1) \end{array}$$

pullback of $f_{n+1}^* \gamma^{n+1}$ along $B_n \to B_{n+1}$ is $f_n^* \gamma^n \oplus \mathbb{R}$.

$\Rightarrow \Sigma^1 Th(f_n^* \gamma^n) \longrightarrow Th(f_{n+1}^* \gamma^{n+1})$.

$\Sigma^1 (MB)_n \longrightarrow (MB)_{n+1}$.

The spectrum represented by these maps is the Thom-spectrum MB.

If (B,f) structure is multiplicative,

Thom spectrum is going to be a ring spectrum:

$$B_{n_1} \times B_{n_2} \longrightarrow B_{n_1+n_2}$$

induce

$$(MB)_{n_1} \wedge (MB)_{n_2} \longrightarrow (MB)_{n_1+n_2}.$$

Pontrjagin - Thom construction gives us a homomorphism

$$\Omega_*^B \to \pi_* MB.$$

Take class $[M] \in \Omega_*^B$. represented by

$$i : M \longrightarrow \mathbb{R}^k.$$

Tubular neighborhood, normal bundle ν of M in $\mathbb{R}^k$

$\nu \cong \underbrace{\text{tubular neighborhood}}_{E_i(M)}.$

$E_i(M) \hookrightarrow \mathbb{R}^k.$

Pontrjagin - Thom collapse.

$$S^k \longrightarrow \underline{Th(\nu)} \cong Th(E_i(M)) \cong \overline{E_i(M)}/\partial\overline{E_i(M)}.$$

$S^k \cong$ one-point compactification of $\mathbb{R}^k$

$$(\mathbb{R}^k)_+ \longrightarrow \overline{E_i(M)}/\partial\overline{E_i(M)}.$$

id on $E_i(M)$, $(\mathbb{R}^k)_+ \setminus E_i(M) \longmapsto$ base point.

Pontrjagin Thom map.

$$S^k \to Th(\nu) \longrightarrow \underbrace{Th(f_{k-n}^* \gamma^{k-n})}_{(MB)_{k-n}}.$$

we end up having a map in $\pi_n MB)$.

<u>Check</u>: homomorphism.

$$PT : \Omega_*^B \to \pi_* MB$$

Thom theorem: $PT: \Omega_*^B \to \pi_* MB$ is an isomorphism.

a step in the proof.

$\{f\} \in \pi_* MB$.

f could be homotoped to a map which is transversal to the zero'th section (regular value) $\rightsquigarrow$ manifold in Ω_*^B.

Real cobordism. .

non-equivariant $\rightsquigarrow$ C_2-equivariant.

C_2 acts by complex conjugation.

Goal: Make MU a C_2-equivariant spectrum.

(Schwede). orthogonal G- spectrum.

Thom spectrum MU.

$MU(n)$ = Thom complex of universal complex n-bundle.

MU: ring spectrum. given by

$MU(n) \wedge MU(m) \longrightarrow MU(n+m)$.

unit maps.

$$S^{\mathbb{C}^n} \to MU(n)$$

$\cong$ Thom space of trivial $\mathbb{C}^n$ over a point.

Observation. These data does not give a C_2-orthogonal ring spectrum.

Define $(MU_{\mathbb{R}})_n = \mathrm{map}(S^{i\mathbb{R}^n}, MU(n))$.

$S^{i\mathbb{R}^n}$ has a C_2-action.

$MU(n)$ has a C_2-action by complex conjugation

$C_2 \curvearrowright \mathrm{map}(S^{i\mathbb{R}^n}, MU(n))$ by conjugation.

$O(n)$ action on complex vector spaces.

$(MU_{\mathbb{R}})_n \wedge (MU_{\mathbb{R}})_m \longrightarrow (MU_{\mathbb{R}})_{n+m}$,

$(f_n, f_m) \longmapsto \mu_{m+n} \circ (f_n \wedge f_m)$

$O(n) \times O(m)$ equivariant.

$f_n : S^{i\mathbb{R}^n} \to MU(n)$, $f_m : S^{i\mathbb{R}^m} \to MU(m)$.

unit. $S^n \longrightarrow (MU_{\mathbb{R}})_n$. defined by adjunction.

$$\mathrm{map}(S^n, \mathrm{map}(S^{i\mathbb{R}^n}, MU(n))) \cong \mathrm{map}(\underbrace{S^{n+i\mathbb{R}^n}}_{S^{n\mathbb{C}}}, MU(n)).$$

$$\begin{tikzcd}
S^1 \wedge (MU_{\mathbb{R}})_n \arrow[r] \arrow[d] & (MU_{\mathbb{R}})_{n+1} \\
(MU_{\mathbb{R}})_1 \wedge (MU_{\mathbb{R}})_n \arrow[ur] &
\end{tikzcd}$$

Check: form a C_2-equivariant orthogonal ring spectrum.

$$\pi_k(MU_{\mathbb{R}}) = \underset{\overrightarrow{n}}{\mathrm{colim}}\, \pi_{k+n}\,\mathrm{map}(S^{i\mathbb{R}^n}, MU(n))$$

$$\overset{\text{adjunction}}{\cong} \underset{\overrightarrow{n}}{\mathrm{colim}}\, \pi_{k+2n} MU(n)$$

$$= \pi_k MU.$$

non-equivariantly, $(MU_{\mathbb{R}})$ is MU.

FACT. $\pi_* MU \cong \mathbb{Z}[x_1, \cdots, x_n, \cdots]$ (by Adams spectral sequence)

$\deg x_i = 2i$

We have defined real cobordism spectrum $MU_{\mathbb{R}}$.

$$\boxed{\Phi^{C_2} MU_{\mathbb{R}} \simeq MO.}$$

(see HHR)

Schwede's notes.

More generally, this method could be used to convert a real spectrum to a C_2-equivariant orthogonal spectrum.

$$\pi_V^{C_2}(MU_{\mathbb{R}}) \cong \operatorname{colim}_n [S^{V+n\rho}, MU(n)]^{C_2}.$$

Def. A multiplicative cohomology theory E, is complex-oriented, if there exists a class

$$x_E \in E^2(\mathbb{C}P^\infty),$$

s.t. its restriction to $E^2(\mathbb{C}P^1)$ is a unit.

Prop. There are computations.

(1) $E^*(\mathbb{C}P^\infty) \cong E^*[[x_E]]$

(2) $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x_E \otimes 1, 1 \otimes x_E]]$

(3). $t: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induces $t^*(x_E)$ is a formal group law over E^*.

Def. Let ξ is a complex n-bundle, E: multiplicative coh thy we say ξ is E-oriented, if there exists $T_\xi \in E^{2n}(B(\xi), S(\xi)) \cong \tilde{E}^{2n}(Th(\xi))$, s.t. its restriction to fiber is a unit. $\tilde{E}^{2n}(S^{2n})$

We say T_ξ is a Thom class.

Prop. If E is complex oriented, then all complex-bundles are E-orientable.

Prop. A ring spectrum E is complex oriented iff there is a ring map

$$MU \to E.$$

MU: complex oriented.

$$x_{MU} : \mathbb{C}P^\infty \xrightarrow{\simeq} MU(1)$$

$$\rotatebox{90}{$\in$}$$

$$MU^2(\mathbb{C}P^\infty).$$

Hill-Hopkins-Ravenel.

Def. Let E be a C_2-equivariant commutative ring spectrum. A Real orientation of E is a class $x_E \in E_{C_2}^{\rho_2}(\mathbb{C}P^\infty)$ whose restriction to $E_2^{\rho_2}(\mathbb{C}P^1)$ is a unit. ($\mathbb{C}P^\infty$ has complex conjugation action).

ρ_2 is the Real regular representation of C_2.

One can check, $\mathbb{C}P^\infty \xrightarrow{\simeq} MU(1)$ C_2-equivalence, gives a Real orientation class in $MU_{\mathbb{R}}^{\rho_2}(\mathbb{C}P^\infty)$.

Thm. E is Real-oriented, w/ Real orientation class x_E.

$$E^\star(\mathbb{C}P^\infty) \cong E^\star[\![x_E]\!]$$

$$E^\star(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^\star[\![x_E \otimes 1, 1 \otimes x_E]\!]$$

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$$

$\rightsquigarrow$ fgl over $E^\star$.

$\rightsquigarrow$ $MU_* \longrightarrow \pi_\star^{C_2} MU_{\mathbb{R}}$. ($MU_*$ universal for fgl's).

$\pi_*^{\bigstar} MU_{\mathbb{R}}$ universal: characterize fgl over graded rings w/ conjugation action. (see HHR for more details).

$\underline{MU^{((G))}} := N_{C_2}^{G} MU_{\mathbb{R}}$

G is a cyclic group C_{2^n}.

norm-forgetful adjunction, gives a map

$$MU^{((H))} \to i_H^* MU^{((G))} \text{ for } H \leq G.$$

Analyze homotopy of $MU^{((C_8))}$.

Equivariant Cobordism.

G = compact Lie group.

<u>Goal</u>. define a G-genuine spectrum MU_G (Lewis-May spectrum)

<u>Construction</u> (tom Dieck).

Let $\mathcal{U}$ be complete universe

$BU_G(n)$ is the classifying space for dim n complex G-vector bundles

$$BU_G(n) = Gr_n(\mathcal{U})$$

n-dim complex subspaces in $\mathcal{U}$.

γ_G^n = universal G-equivariant n-bundle.

As usual, we regard a complex representation as a bundle over a point.

Define a prespectrum, D,

$D_V = Th(\gamma^{|V|}_G)$. $V \subseteq \mathcal{U}$

$|V|$ = complex dimension of V.

$V \subset W \subset \mathcal{U}$.

classifying map gives a structure map

$$\Sigma^{W-V} D_V \to D_W.$$

We spectrify to get a genuine G-spectrum

MU_G (homotopical complex cobordism).

Rmk. We could define "complex" G-manifolds, and there is a ring $\Omega^{U,G}_*$ of cobordism classes of them. There is still a Pontryagin-Thom construction. (geometric complex cobordism).

$M \to \mathbb{R}^V$

Pontryagin Thom Construction.

$$S^V \to Th(\gamma_G^{\cdot\cdot}). \quad \pi(MU_G).$$

This is a ring homomorphism, but not an isomorphism.

Prop. MU_G is a ring spectrum.

The htpy group of MU_G is graded by $\mathbb{Z}$.

Take a complex representation V of dimension n, classifying map induces

$$S^V \to Th(\gamma^G_n)$$

represents an element $\tau_V \in MU_G^{2n}(S^V)$. is called the Thom class of representation V.

Prop. τ_V is invertible in $MU_G^{\star}$, hence it induces an isomorphism.

$$\sigma_V: \ MU_G^{-V}(X) \longrightarrow MU_G^{-2|V|}(X).$$

Pf $\tau_V: S^V \to Th(\gamma_G^n)$ has an inverse given by $S^{2n} \to Th(\gamma_G^{|V|}) \ldots$

$$\begin{array}{c} Th(\gamma_G^{|V|}) \\ \cong \\ \gamma_G^n \end{array}$$

classifying map for $\mathbb{C}^n$, identify it w/ $S^{2n} \to Th(\gamma_G^{|V|})$.

smash them is nullhomotopic, so τ_V is invertible.

This proves that MU_G is a complex stable cohomology theory.

Def. Let E_G be a $R(G)$-graded cohomology theory, if there are $\lambda(V) \in E_G^{2|V|}(S^V)$, s.t. multiplication gives isomorphism

$$\sigma_V: \ \underset{E_G^{n+V}(S^V\wedge X)}{\overset{\shortparallel}{E_G^n(X)}} \longrightarrow \underset{E_G^{n+2|V|}(S^V \wedge X)}{\overset{\shortparallel}{E_G^{n+2|V|}(S^V\wedge X)}},$$

then we say E_G is complex stable. V = complex rep of G.

$\Rightarrow \pi_\star^G E_G$ is $\mathbb{Z}$-graded.

Ex/. MU_G, equivariant K-theory, Borel cohomology theory.

Def. A G-equivariant multiplicative (complex stable) cohomology theory is said to have natural Thom classes, if for every complex G-vector bundle ξ over X, of complex dimension n, there exists a Thom class $\tau_\xi \in E^{2n}(Th(\xi))$, s.t.

① . naturality . If $f : Y \to X$ is an equiv map,

then $\tau_{f^*\xi} = f^* \tau_\xi$

② multiplicativity . If ξ, η are vector bundles,

then $\tau_{\xi \times \eta} = \tau_\xi \wedge \tau_\eta$.

③ . normalization . If V is a representation ,

then $\tau_V = \sigma_V(1)$.

$$\sigma_V : E_G^0(S^0) \xrightarrow{\sigma_V} E_G^{2|V|}(S^V) .$$

There exists universal Thom classes

$$\tau_n \in E_G^{2n}(Th(\gamma_n^G)) \text{ for each } n$$

+ some conditions.

$\rightsquigarrow$ Thom classes for all bundles.

Def . Let A be an abelian compact Lie group.

We say a multiplicative A-equiv cohomology theory is complex oriented, if there exists a cohomology class

$$x_A \in E_A^*(\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\varepsilon))$$

that restricts to a unit in

$$E_A^*(\underbrace{\mathbb{C}P(\alpha \oplus \varepsilon), \mathbb{C}P(\varepsilon)}_{S^{\alpha^{-1}}}) \text{ for all linear}$$

characters of A .

$$\overset{\cong}{E_A^*(S^{\alpha^{-1}})}$$

$x_A \in E^2(\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\varepsilon))$

Thm. If E is a complex oriented theory , then all A-equivariant complex vector bundles are E-orientable ($\exists$ Thom classes) .

Sketch . $x_A \in E_A^2(\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\varepsilon))$.

restrict to a class $y_A \in E_A^2(\mathbb{C}P(\mathcal{U}))$.

One can compute that

$$\widetilde{E}_A^*(BU_A(n)) \cong \left(\widetilde{E}_A^*(\mathbb{C}P(\mathcal{U}))^{\otimes n}\right)^{\Sigma_n}$$

$y_A \in E_A^*(\mathbb{C}P(\mathcal{U}))$

$y_A^{\otimes n}$ gives the desired Thom class.

$$\widetilde{E}_A^*(\underbrace{BU_A(n), \cancel{BU_A}*1}_{\text{Cellular structure}}) \cong \widetilde{E}_A^*(Th(\gamma_n^G)).$$

<u>Thm</u>. A complex-orientation of E_A is the same as a ring map $MU_A \to E_A$.

<u>Talk on ~~Wednes~~ Saturday</u> complex-oriented A-equivariant theory $\longrightarrow$ A-equivariant formal group law. ...

Some other properties and results for MU_G.

- MU_G is a split spectrum. (MU_G has underlying non-equivariant spectrum MU).
- For finite groups, it has localization and completion theorems.

$$F(EG_+, MU_G) \simeq (MU_G)^{\wedge}_{\text{augmentation ideal}}$$

$MU_G \to MU$.

- There are some computations for MU_G^*.

$\pi_* MU$

$\pi_*^{C_2} MU_{\mathbb{R}}$

$\pi_*^G MU_G$

Isotropy separation. For example $G = \mathbb{Z}/p$.

$$E\mathbb{Z}/p_+ \to S^0 \to \widetilde{E\mathbb{Z}/p}$$

Smash w/ $MU_{\mathbb{Z}/p}$.

On htpy groups, there is a pullback diagram.

$$\begin{array}{ccccc} E\mathbb{Z}/p_+ \wedge MU_{\mathbb{Z}/p} & \longrightarrow & MU_{\mathbb{Z}/p} & \longrightarrow & MU_{\mathbb{Z}/p} \wedge \widetilde{E\mathbb{Z}/p} \\ \downarrow \sim & & \downarrow & & \downarrow \checkmark \\ E\mathbb{Z}/p_+ \wedge F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p}) & \longrightarrow & F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p}) & \longrightarrow & F(E\mathbb{Z}/p_+, MU) \wedge \widetilde{E\mathbb{Z}/p} \end{array}$$

✓

pullback square.

$\pi_* MU_{\mathbb{Z}/p}$ is pull back of this diagram.

We compute the three corners of the pullback square

$$\pi_* F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p})^{\mathbb{Z}/p} \cong MU_*(B\mathbb{Z}/p).$$

$$= MU_*[[x]] / [p]_F x.$$

Top right, is the geometric fixed point $\underline{\Phi^{\mathbb{Z}/p} MU_{\mathbb{Z}/p}}$

Compute $\mathbb{Z}/p$-fixed point of Thom spaces.

Bottom map is localization map.

Combining these we have result for

$\pi_* MU_{\mathbb{Z}/p}$.