The homotopy of $H\mathbb{Z}$

28. July 2021

- How does htpy of $H\underline{\mathbb{Z}}$ fit into the proof?
- What is $H\underline{\mathbb{Z}}$?
- How do we compute it?

HHR: Kervaire invariant one problem.

KO $\quad S^0 \longrightarrow KO$

$\eta \in \pi_1 S^0 \longmapsto$ nontrivial

Ω $\quad S^0 \longrightarrow \Omega$

Kervaire inv. 1 elements $\longmapsto$ nontrivially.

$$\Omega := (D^{-1} N_{C_2}^{C_8} MU_{\mathbb{R}})^{C_8}$$

$MU_{\mathbb{R}}$: Real bordism
$C_2 \curvearrowright$ [illegible] conjugate

Try to attack $N_{C_2}^{C_8} MU_{\mathbb{R}}$.

Slice spectral sequence:
it computes homotopy of $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$

E_2: computed from $\underline{\pi}_\star H\underline{\mathbb{Z}}$.

$H\underline{\mathbb{Z}}$: Equivariant Eilenberg—Mac Lane spectrum of the Mackey functor $\underline{\mathbb{Z}}$.

Mac Lane v.s. MacLane

he changed his name. former name

Eilenberg-Mac Lane spectrum:

$A \in Ab \quad HA \rightsquigarrow \pi_0 HA \cong A$

$\pi_i HA \cong 0. \quad i \neq 0$

equiv. Eilenberg-Mac Lane spectrum:

Mackey functor $\ni \underline{M}$

$H\underline{M} \rightsquigarrow \underline{\pi}_0 H\underline{M} \cong \underline{M}$

$\underline{\pi}_i H\underline{M} \cong \underline{0} \quad i \neq 0 \quad i \in \mathbb{Z}.$

In general $\underline{\pi}_V H\underline{M} \ncong \underline{0}$.

Thm (Lewis-May-McClure) [Hard to read]

HM̲ is uniquely defined up to homotopy.

Ref: Lewis. Equivariant Eilenberg-Mac Lane spaces
and the equivariant Seifert-Van Kampen thm.
and suspension theorem.

Transfer: only stable.

only use its axioms.

back to Hatcher.

Mackey functor v.s. Abelian groups.

G finite gp.

$B_G := \Big\{$ obj. finite G-sets

mor. $\Big\{$

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

$\swarrow \searrow$ ~ maps of G-sets

$\Big/ \sim \Big($

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] \arrow[d, "\cong"] & \\ X & Z' \arrow[l] \arrow[r] & Y \end{tikzcd}$$

$\Big)\Big\}$.

group completion.

unit

$$\begin{tikzcd} & \emptyset \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

addition: $\underline{\amalg}$.

Composition:

$$\begin{tikzcd} & & Z \arrow[dl] \arrow[dr] & & \\ & Z_1 \arrow[dl] \arrow[dr] & & Z_2 \arrow[dl] \arrow[dr] & \\ X & & Y & & W \end{tikzcd}$$

B_G: enriched over Ab.

$\mathrm{Mack}_G : \mathrm{Fun}_{add}(B_G^{op}, Ab)$

Day convolution: Mack_G is a closed sym. monoidal cat.

$\underline{M} \square \underline{N}$ is a Mackey functor.

$\mathrm{Hom}(\underline{M}, \underline{N})$ is a Mackey functor.

$H \leq G$

$\uparrow_H^G : \mathrm{Mack}_H \longrightarrow \mathrm{Mack}_G$

$\underline{M} : B_H^{op} \longrightarrow Ab \longmapsto$

$$\begin{tikzcd} B_G^{op} \arrow[d, "i_H^*"] \arrow[dr, "\uparrow_H^G \underline{M}"] & \\ B_H^{op} \arrow[r, "\underline{M}"] & Ab \end{tikzcd}$$

$\underline{\pi}_\star H\underline{\mathbb{Z}}$ i.e. $V \in RO(G) \leftarrow$ orthogonal rep. ring

$[S^V, H\underline{\mathbb{Z}}]^G$.

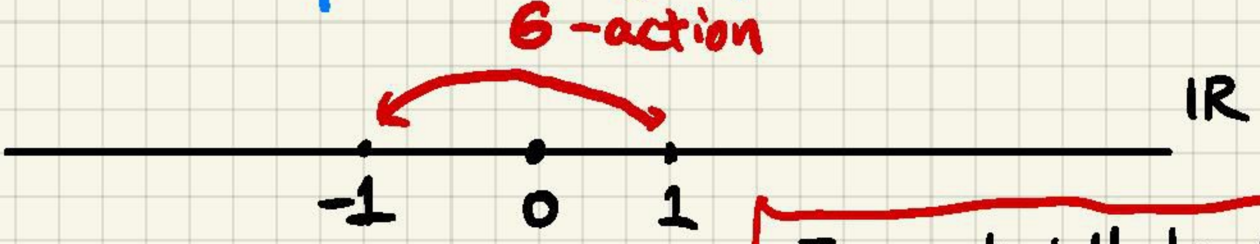
$G = \{*\}$. $RO(G) = \mathbb{Z}$.

$$[S^n, H\mathbb{Z}] \cong \pi_n H\mathbb{Z} = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

$G = C_2$. C_2: cyclic group of order 2.

$$RO(C_2) = \underset{\{1\}}{\mathbb{Z}} \oplus \underset{\{\sigma\}}{\mathbb{Z}}$$

trivial rep. sign rep.



$\underline{\mathbb{Z}}$: constant Mackey functor

$\underline{\mathbb{Z}}(G/H) \cong \mathbb{Z}$

all restrictions are id.
$K \subseteq H \subseteq G$

$Tr_K^H : \mathbb{Z} \xrightarrow{[H:K]} \mathbb{Z}$

$$[S^V, H\underline{\mathbb{Z}}]^G$$

$V = a + b\sigma \quad a, b \in \mathbb{Z}$.

$$[S^{a+b\sigma}, H\underline{\mathbb{Z}}]^G$$

E a spectrum/homology theory

$$[X, E]_* \cong E^*(X) \qquad [S^0, X \wedge E]_* \cong E_*(X)$$

$$[S^{a+b\sigma}, H\underline{\mathbb{Z}}]^G_* \cong \tilde{H}^*_G(S^{a+b\sigma}; \underline{\mathbb{Z}})$$

singular version: $G/H \times \Delta^n$.

$$[S^{a+b\sigma}, H\underline{\mathbb{Z}}]^G_* \cong [S^a, S^{-b\sigma} \wedge H\underline{\mathbb{Z}}]^G_*$$

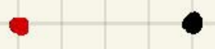
I mean ab. gps as $\underline{(-)}(G/G)$

$$\begin{tikzcd} {[S^a, S^{-b\sigma} \wedge H\underline{\mathbb{Z}}]^G_*} \arrow[d, phantom, "\cong" rotate=90] \\ \tilde{H}^G_a(S^{-b\sigma}; \underline{\mathbb{Z}}) \end{tikzcd}$$

$b \leq 0$ so that $-b \geq 0$.

$b=0$: $\underline{\tilde{H}}_*(S^0; \underline{\mathbb{Z}})$

S^0 :



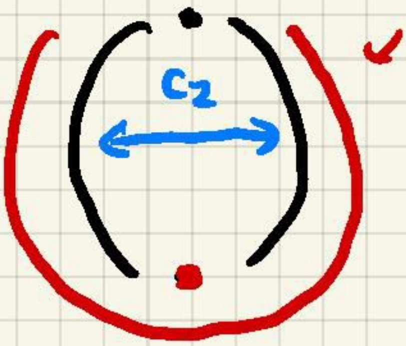
non-eq: $\begin{matrix} 0 \\ \mathbb{Z} \end{matrix}$

eq: $\begin{matrix} 0 \\ \underline{\mathbb{Z}} \end{matrix}$

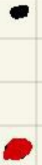
$$\tilde{H}_*(S^0; \mathbb{Z}) = \begin{cases} \mathbb{Z} & *=0 \\ \underline{0} & * \neq 0. \end{cases}$$

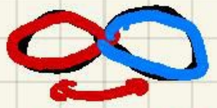
one-point-compactification.

$b=1$: $\underline{\tilde{H}}_*(S^\sigma; \underline{\mathbb{Z}})$

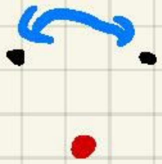


$\mathbb{R}$ with C_2-action.

0-cell: 

1-cell: 

$S^1 \wedge C_{2+}$ $\Uparrow$ reduced suspension

C_{2+}: 

cellular chain complexes:

0	1
$\underline{\mathbb{Z}}$	?

$$? \cong \underline{\pi}_1(S^1 \wedge C_{2+} \wedge H\underline{\mathbb{Z}})$$

$$\cong \underline{\pi}_0(C_{2+} \wedge H\underline{\mathbb{Z}})$$

Mack_G

Wirthmüller iso. $\Longrightarrow$ $\underline{\pi}_V(G_+ \wedge_H X)$ $V \in RO(G)$

$X \in Sp^H$.

$$\cong$$

$$\uparrow_H^G(\underline{\pi}_{i_H^* V}(X))$$ in Mack_H

Wirthmüller
$\downarrow$

$$? \cong \underline{\pi}_0(C_{2+} \wedge H\underline{\mathbb{Z}}) \cong \uparrow_e^{C_2} \pi_0(H\mathbb{Z}) \cong \uparrow_e^{C_2} \mathbb{Z}.$$

exercise: $\uparrow_e^{C_2}\mathbb{Z} \cong \underline{\mathbb{Z}[C_2]}$: fixed points Mackey functor of $\mathbb{Z}[C_2] \leftsquigarrow C_2$-module.

Lewis diagram: C_2 : $\underline{M} \in \mathrm{Mack}$

$$\begin{tikzcd}
C_2/C_2 & \underline{M}(C_2/C_2) \arrow[d, bend right, "\mathrm{Res}"'] \\
C_2/e & \underline{M}(C_2/e) \arrow[u, bend right, "\mathrm{Tr}"']
\end{tikzcd}$$

M is a G-module, the fixed pt. MF. $\underline{M}$:

$\underline{M}(G/H) := M^H$

Res : inclusion

Tr : summation over H/K

$C_2 = \langle \gamma \rangle$

$$\begin{tikzcd}
\underline{\mathbb{Z}} : & \mathbb{Z} \arrow[d, bend right, "1"'] & & \underline{\mathbb{Z}[C_2]} : & \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\
& \mathbb{Z} \arrow[u, bend right, "2"'] & & & \mathbb{Z}[C_2] \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$

$$\begin{aligned}
\Delta : \mathbb{Z} &\longrightarrow \mathbb{Z}[C_2] \\
1 &\longmapsto 1+\gamma
\end{aligned}$$

$$\begin{aligned}
\nabla : \mathbb{Z}[C_2] &\longrightarrow \mathbb{Z} \\
1, \gamma &\longmapsto 1.
\end{aligned}$$

Chain cplx for S^σ :

$$\begin{tikzcd}
& 2 & 1 \arrow[l, mapsto] & \\
& 0 & 1 & \\
1 \arrow[d, mapsto] & \mathbb{Z} \arrow[d, bend right, "1"'] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[d, bend right, "\Delta"'] & 1 \arrow[d, mapsto] \\
1 & \mathbb{Z} \arrow[u, bend right, "2"'] & \mathbb{Z}[C_2] \arrow[l, two heads, "\nabla"'] \arrow[u, bend right, "\nabla"'] & 1+\gamma \\
& 2 & 1+\gamma \arrow[l, mapsto] &
\end{tikzcd}$$

$\rightsquigarrow$ compute $\hat{H}_*(\underline{S};\underline{\mathbb{Z}})$

homology:

$$\begin{array}{cc} 0 \quad \overset{\langle a_6 \rangle}{\diagup\!\!\diagup} & 1 \\ \mathbb{Z}/2 & 0 \\ & \\ 0 & \mathbb{Z}_- \end{array}$$

$\mathbb{Z}$ with $\{-1\}$-action.

$$\underline{\tilde{H}}_*(S^6;\underline{\mathbb{Z}}) \cong \begin{cases} \begin{tikzcd} \mathbb{Z}/2 \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd} & * = 0 \\ \begin{tikzcd} 0 \arrow[d, bend right] \\ \mathbb{Z}_- \arrow[u, bend right] \end{tikzcd} & * = 1 \\ \underline{0} & * \neq 0,1,\ * \in \mathbb{Z}. \end{cases}$$

- Why $\underline{\pi}_V(X)$ is a Mackey functor?

  Ref: Hill-Hopkins-Ravenel after Def. 3.1.

- What is the internal hom of Mackey functors?

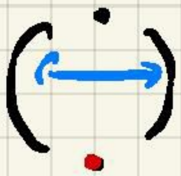
  Ref: Lewis. The theory of Green functors, first few pages.

- Why $\mathbb{Z}_-$ in $\underline{\tilde{H}}_1$?

Answer $\underline{1}$: $\mathbb{Z} \xleftarrow{\nabla} \mathbb{Z}[C_2]$ $\quad C_2 \curvearrowright \langle 1-\gamma \rangle$

$\ker(\nabla) = \langle 1-\gamma \rangle$ $\quad 1 \mapsto \gamma \cdot \gamma \mapsto 1$

$1-\gamma \mapsto \gamma - 1$

Answer 2:  C_2-action is orientation-reversing

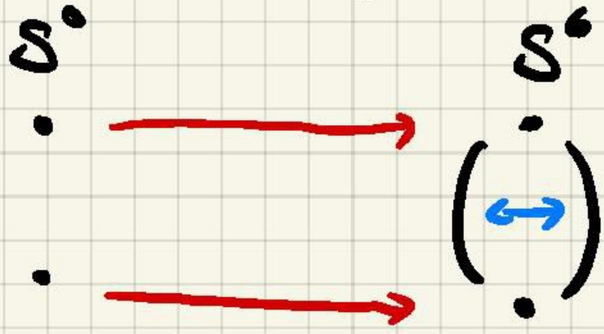
Def. $a_6 \in \underline{\tilde{H}}_0(S^6;\underline{\mathbb{Z}})(G/G)$ is the generator.

$$\begin{tikzcd} \underline{\tilde{H}}_0(S^6;\underline{\mathbb{Z}})(G/G) \arrow[d, phantom, "\cong"] & \\ {[S^0, S^6\wedge H\underline{\mathbb{Z}}]} \arrow[r, phantom, "\cong"] & {[S^{-6}, H\underline{\mathbb{Z}}]} \arrow[d, phantom, "\cong"] \\ & \underline{\pi}_{-6}(H\underline{\mathbb{Z}}) \end{tikzcd}$$

$H\underline{\mathbb{Z}}$ is a ring spectrum: $S^0 \longrightarrow H\underline{\mathbb{Z}}$.

$\underline{\pi}_{-6}S^0 \longrightarrow \underline{\pi}_{-6}H\underline{\mathbb{Z}}$. (Hurewicz map)

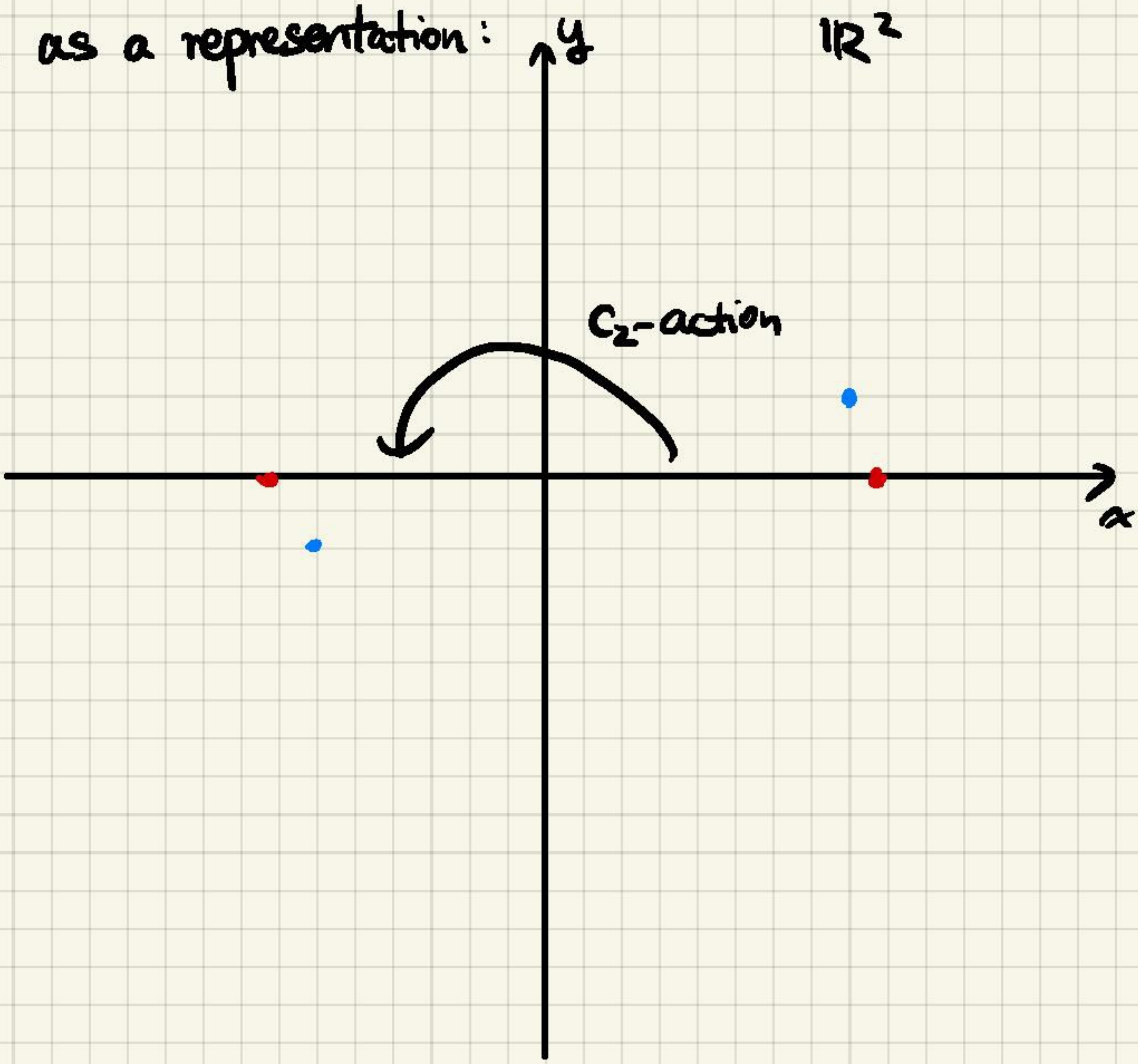
- a_6 is in the image:



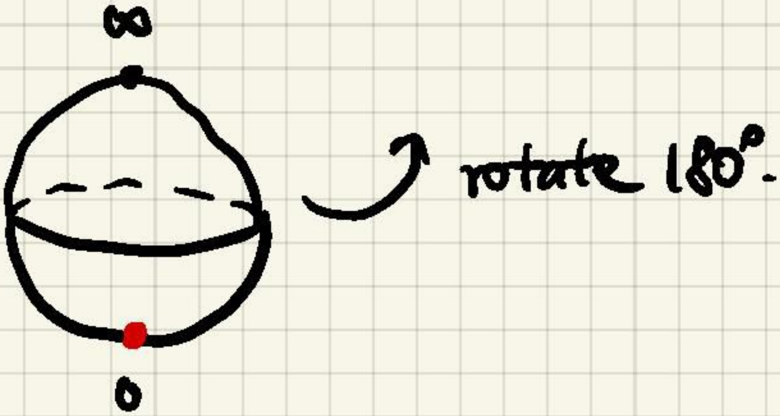
This is a_6.

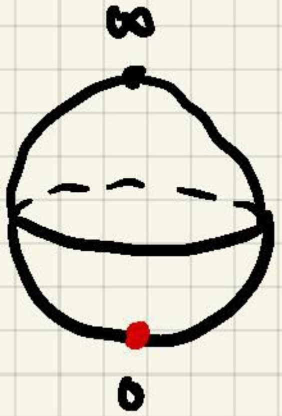
$b=2$: $\underline{\tilde{H}}_0(S^{26};\underline{\mathbb{Z}})$.

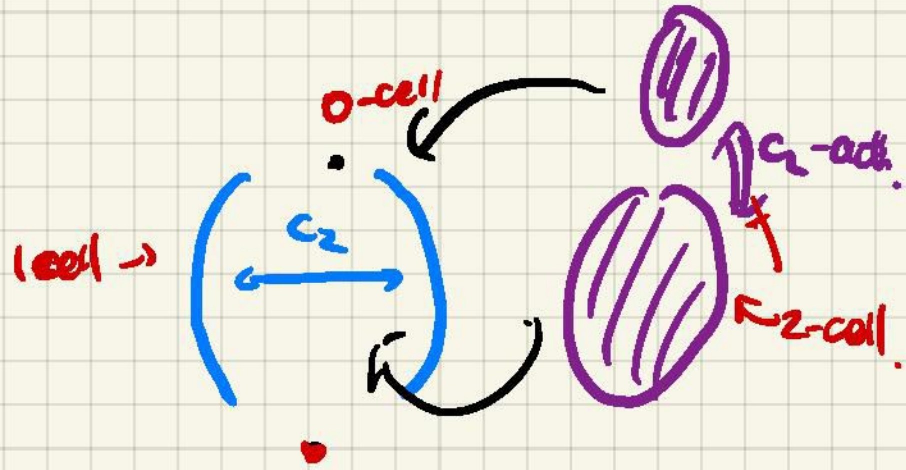
2σ as a representation:



$\Downarrow$ one-point-compactification







cellular chain cplx for $S^{2\sigma}$:

$$\begin{tikzcd}
0 & 1 & 2 \\
\mathbb{Z} \arrow[d, bend right, "1"'] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z} \arrow[u, bend right, "2"'] & \mathbb{Z}[C_2] \arrow[l, "\nabla"'] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}[C_2] \arrow[l, "1-\gamma"'] \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$

$$\begin{tikzcd}
1 & 1,\ \gamma \arrow[l] & 1-\gamma & 1 \arrow[l] \\
 & & \gamma-1 & \gamma \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
0 & 1 & 2 \\
\mathbb{Z}/2 = \langle a_\sigma^2 \rangle & 0 & \mathbb{Z} = \langle u_{2\sigma} \rangle \arrow[d, bend right, "1"'] \\
0 & 0 & \mathbb{Z} \arrow[u, bend right, "2"']
\end{tikzcd}$$

$$\underline{\tilde{H}}_*(S^{2\sigma};\underline{\mathbb{Z}}) \cong \begin{cases} \begin{tikzcd} \mathbb{Z}/2 \arrow[d, bend right] \\ 0 \arrow[u, bend right, "2"'] \end{tikzcd} & * = 0 \\ \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd} & * = 2 \\ \underline{0} & * = \text{otherwise}. \end{cases}$$

In general : $\underline{\tilde{H}}_*(S^{n\sigma};\underline{\mathbb{Z}})$ for $n \geq 0$

is generated by a_σ and $u_{2\sigma}$- powers.

e.g. $\underline{\widehat{H}}_*(S^{3\sigma};\underline{\mathbb{Z}})$

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \mathbb{Z}/2 = \langle a_\sigma^3 \rangle & & \mathbb{Z}/2 = \langle u_{2\sigma} a_\sigma \rangle & 0 \\ & \underline{0} & & \\ 0 & & 0 & \mathbb{Z}. \end{array}$$

$\underline{\widehat{H}}_*(S^{4\sigma};\underline{\mathbb{Z}})$

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ \mathbb{Z}/2 = \langle a_\sigma^4 \rangle & & \mathbb{Z}/2 = \langle u_{2\sigma} a_\sigma^2 \rangle & & \mathbb{Z} = \langle u_{2\sigma}^2 \rangle \\ & \underline{0} & \cdot & \underline{0} & {}^{1}\!\downarrow\uparrow^{2} \\ 0 & & 0 & & \mathbb{Z} \end{array}$$

Thm. $\underline{\hat{H}}_\star(S^{n\sigma};\underline{\mathbb{Z}})$ forms a ring.

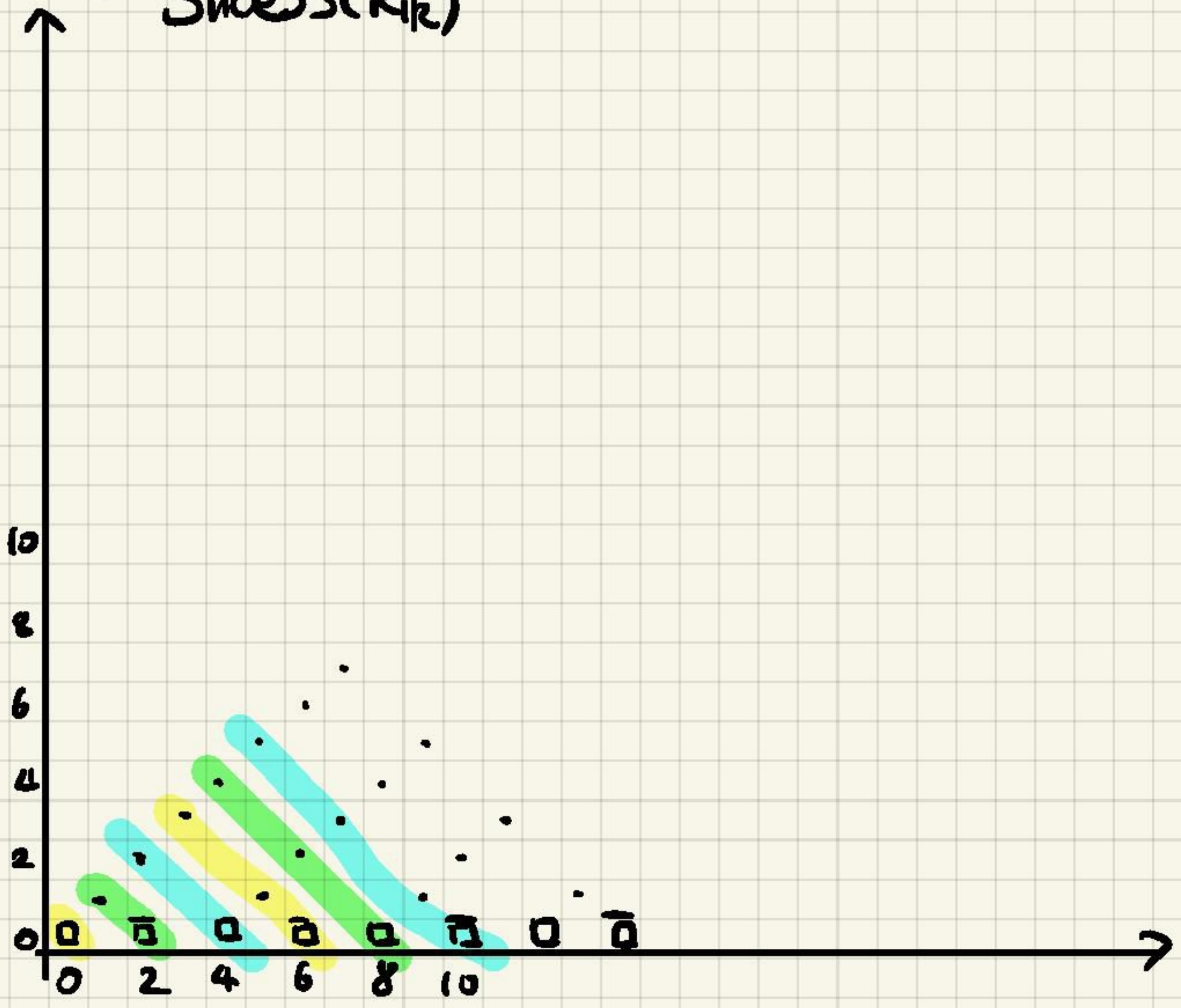
$$\begin{gathered} \cong \\ \mathbb{Z}[u_{2\sigma}, a_\sigma]/(2a_\sigma = 0). \end{gathered}$$

Ningchuan's talk: SliceSS($K_{\mathbb{R}}$)

sliceSS($k_{\mathbb{R}}$) $k_{\mathbb{R}} := \tau_{\geq 0} K_{\mathbb{R}}$.

$$\bullet := \begin{tikzcd} \mathbb{Z}/2 \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd} \qquad \square := \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd} \qquad \overline{\square} := \begin{tikzcd} 0 \arrow[d, bend right] \\ \mathbb{Z}_- \arrow[u, bend right] \end{tikzcd}$$

SliceSS($k_{\mathbb{R}}$)



1st diagonal: $\underline{\hat{H}}_*(S^0;\underline{\mathbb{Z}})$

2nd diagonal: $\underline{\hat{H}}_*(S^6;\underline{\mathbb{Z}})$

3rd diagonal: $\underline{\hat{H}}_*(S^{26};\underline{\mathbb{Z}})$

$\vdots$

$\underline{\pi}_\star H\underline{\mathbb{Z}}$ gives E_2-page of the Slice spectral sequence.

- Multiplication in $H\underline{\mathbb{Z}}$:

R is a ring. HR is a ring spectrum.

$HR \wedge HR \xrightarrow{m} HR$ is given by

$R \otimes R \longrightarrow R$

$\underline{\mathbb{Z}}$ is a Mackey functor. $\underline{\mathbb{Z}} \square \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}$

$$\Downarrow$$

$H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}} \xrightarrow{m} H\underline{\mathbb{Z}}$.

The dark side:

- How about SliceSS$(K_{\mathbb{R}})$

$\underline{\hat{H}}_*(S^{n6};\underline{\mathbb{Z}})$ for $n<0$. ???

D: S-W duality.

$$\tilde{H}_*(X;\mathbb{Z}) \cong \tilde{H}^*(DX;\mathbb{Z})$$

for $n<0$ $\quad \hat{\underline{H}}_*(S^{n\sigma};\underline{\mathbb{Z}}) \cong \tilde{\underline{H}}^*(S^{-n\sigma};\underline{\mathbb{Z}})$.

From cellular homology to cellular cohomology:

take $\mathrm{Hom}(-,\mathbb{Z})$.

Equivariantly:

$\hat{\underline{H}}^*(S^{n\sigma};\underline{\mathbb{Z}})$ for $n \geq 0$.

$n=0$: $\quad \hat{\underline{H}}^*(S^0;\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}} & *=0 \\ 0 & *=\text{other.} \end{cases}$

$n=1$: $\quad \hat{\underline{H}}^*(S^\sigma;\underline{\mathbb{Z}})$:

Chain cplx for S^σ:

$$\begin{tikzcd}
 & 2 & & 1 \arrow[ll, mapsto] & \\
 & 0 & & 1 & \\
1 \arrow[d, mapsto] & \mathbb{Z} \arrow[d, bend right, "1"'] & & \mathbb{Z} \arrow[ll, "\times 2"'] \arrow[d, bend right, "\Delta"'] & 1 \arrow[d, mapsto] \\
1 & \mathbb{Z} \arrow[u, bend right, "2"'] & & \mathbb{Z}[C_2] \arrow[ll, two heads, "\nabla"'] \arrow[u, bend right, "\nabla"'] & 1+\gamma \\
 & 2 & & 1+\gamma \arrow[ll, mapsto] &
\end{tikzcd}$$

$\rightsquigarrow$ compute $\hat{H}_*(S^\sigma;\mathbb{Z})$

$\underline{\mathrm{Hom}}(-, \underline{\mathbb{Z}})$ ⇝ is also a Mackey functor!!!

Lemma. $\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$

$\underline{\mathrm{Hom}}(\underline{\mathbb{Z}[C_2]}, \underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}[C_2]}$.

cochain complex for S^σ

$$\begin{tikzcd} 0 & & 1 \\ \mathbb{Z} \arrow[rr, "1"] \arrow[d, bend right, "1"'] & & \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\ \mathbb{Z} \arrow[rr, "\Delta"] \arrow[u, bend right, "2"'] & & \mathbb{Z}[C_2] \arrow[u, bend right, "\nabla"'] \end{tikzcd}$$

cohomology:

$$\begin{tikzcd} 0 & & \underline{1} \\ & & 0 \\ \underline{0} & & \mathbb{Z}_- \end{tikzcd}$$

$$\underline{\widehat{H}}^*(S^\sigma; \underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}} & * = 1 \\ \underline{0} & * = \text{otherwise} \end{cases}$$

The dual of a_σ is NOT here.

$\Downarrow$

The gap theorem.

$n=2$: $\underline{\tilde{H}}^*(S^{2\sigma}; \underline{\mathbb{Z}})$

cellular chain cplx for $S^{2\sigma}$:

$$\begin{tikzcd}
0 & & 1 & & 2 \\
\mathbb{Z} \arrow[d, bend right, "1"'] & & \mathbb{Z} \arrow[ll, "\times 2"'] \arrow[d, bend right, "\Delta"'] & & \mathbb{Z} \arrow[ll, "0"'] \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z} \arrow[u, bend right, "2"'] & & \mathbb{Z}[C_2] \arrow[ll, "\nabla"'] \arrow[u, bend right, "\nabla"'] & & \mathbb{Z}[C_2] \arrow[ll, "1-\gamma"'] \arrow[u, bend right, "\nabla"'] \\
1 & & \begin{smallmatrix} 1 \\ \gamma \end{smallmatrix} \arrow[ll, mapsto] & 1-\gamma & 1 \arrow[l, mapsto] \\
 & & & \gamma-1 & \gamma \arrow[l, mapsto]
\end{tikzcd}$$

cochain cplx :

$$\begin{tikzcd}
0 & & 1 & & 2 \\
\mathbb{Z} \arrow[rr, "1"] \arrow[d, bend right, "1"'] & & \mathbb{Z} \arrow[rr, "0"] \arrow[d, bend right, "\Delta"'] & & \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z} \arrow[rr, "\Delta"] \arrow[u, bend right, "2"'] & & \mathbb{Z}[C_2] \arrow[rr] \arrow[u, bend right, "\nabla"'] & & \mathbb{Z}[C_2] \arrow[u, bend right, "\nabla"'] \\
 & & 1 \arrow[rr, mapsto] & & 1-\gamma \\
 & & \gamma \arrow[rr, mapsto] & & \gamma-1
\end{tikzcd}$$

cohomology:

$$\begin{tikzcd} 0 & 1 & 2 \\ & & \mathbb{Z} = \langle \theta \rangle \arrow[d, bend right, "2"'] \\ \underline{0} & \underline{0} & \mathbb{Z} \arrow[u, bend right, "1"'] =: \boxslash \end{tikzcd}$$

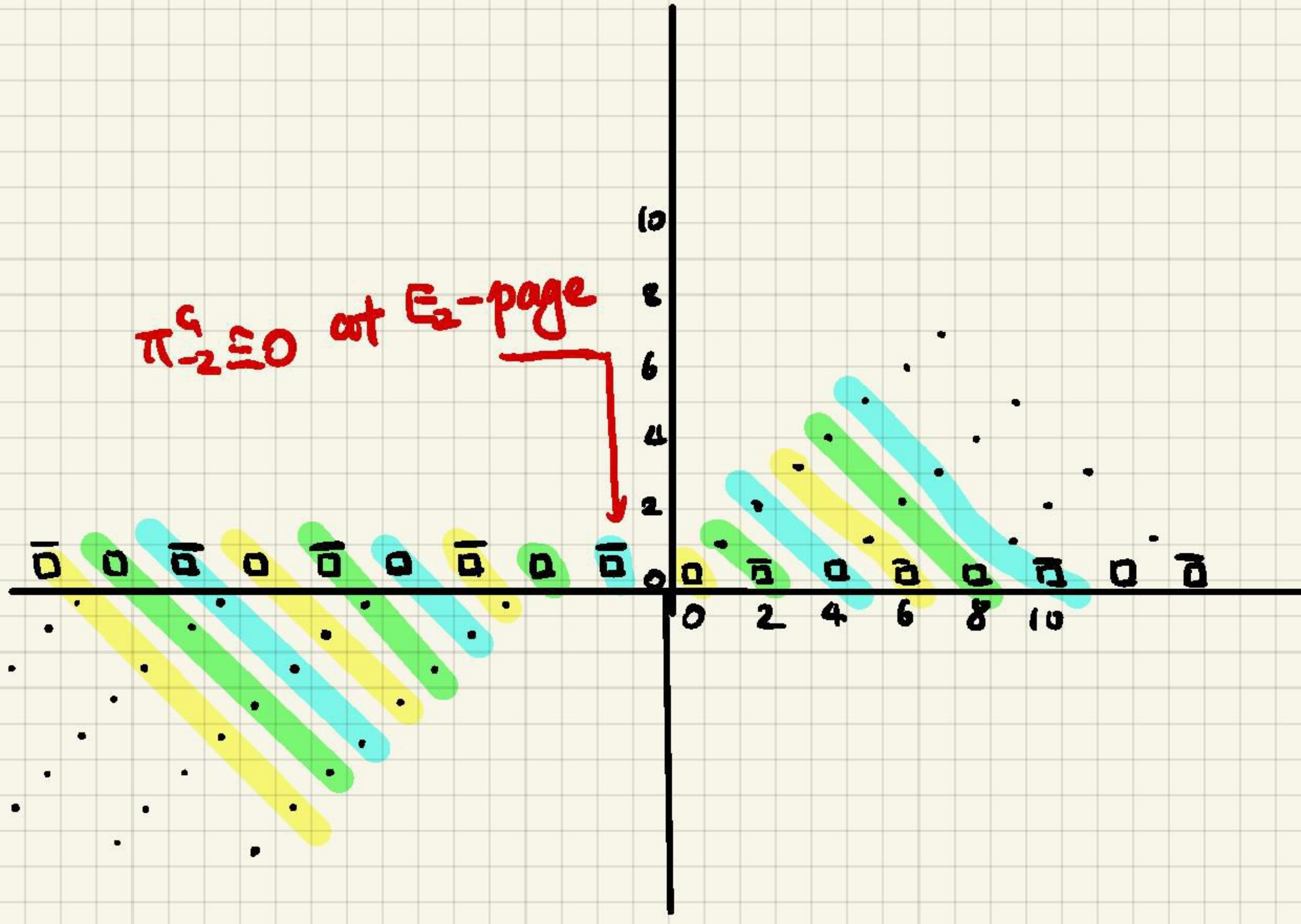
Thm (algebraic Gap thm).

$G = C_{2^n}$. V is an actual representation.

$$\underline{\tilde{H}}^*(S^V; \underline{\mathbb{Z}}) = \underline{0} \text{ for } * = 0, 1.$$

How to get the gap theorem?

SliceSS$(K_{\mathbb{R}})$:



$\pi^G_{-2} \cong 0$ at E_2-page