

# The Detection Theorem

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30. July 2021

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# The detection thm.

Review:  $\cdot MU^{(C_2)} := N_{C_2}^{C_2} MU_{\mathbb{R}}$ .

$$D \in \pi_{19}^{C_2} MU^{(C_2)} \quad D = N_2^8(E_1^{C_2}) N_2^4(E_2^{C_2}) N_2^8(E_8^{C_2})$$

$\downarrow$

$$D^{-1} MU^{(C_2)} \rightsquigarrow \Omega := (D^{-1} MU^{(C_2)})^{C_2}$$

Properties of  $\Omega$ :  $\cdot \pi_{-2} \Omega \cong 0$  The gap thm.

[Slice tower of  $MU^{(C_2)}$

$$+ \pi_{-2}^{C_2} \Sigma^{i/e} H\mathbb{Z} \cong 0]$$

$\cdot \pi_{256+r} \Omega \cong \pi_r \Omega$ .

[Slice differentials + HFP Thm]

Thm (Detection) Let  $\theta_j \in \pi_{2^{j+1}-2} \mathcal{S}$  be an element of Kervaire invariant 1,  $j > 3$ , then

image of  $\theta_j$  in  $\pi_{2^{j+1}-2} \Omega$  is non-zero.

$\Omega$  is a ring spectrum:  $MU^{(C_2)}, D^{-1} MU^{(C_2)}$   $G$ -com. rifs.  
 $( )^{C_2}$  gets ring spectrum.

$$\mathcal{S} \longrightarrow \Omega \rightsquigarrow \text{take } \pi_{2^{j+1}-2} (-).$$

$\text{Cor(HHR)} \theta_j$  for  $j \geq 7$  doesn't exist.

Questions: • Why  $G = C_8$ ?

• Why sth. like this can be true?

\* The dark force of chromatic homotopy theory!

Origin: Ref. K-M. Groups of homotopy spheres, I.

Thm (Kervaire-Milnor) group of smooth structures on  $S^n$ .

$$\begin{array}{ccccccc}
 n \not\equiv 2 \pmod{4} & & & & & & \\
 & & \downarrow & & \text{cokernel of } \mathcal{J} \text{ homomorphism} & & \\
 0 \longrightarrow & \theta_n^{\text{bp}} & \longrightarrow & \theta_n & \longrightarrow & \pi_n/\mathcal{J} & \longrightarrow 0 \\
 & \uparrow & & & & \downarrow & \\
 & \text{subgp. that bounds a parallelizable mfd.} & & & & & 
 \end{array}$$

$n \equiv 2 \pmod{4}$

$$\begin{array}{ccccccc}
 0 \longrightarrow & \theta_n^{\text{bp}} & \longrightarrow & \theta_n & \longrightarrow & \pi_n/\mathcal{J} & \xrightarrow{\Phi} \mathbb{Z}/2 \longrightarrow \theta_{n-1}^{\text{bp}} \longrightarrow 0 \\
 & & & & & \uparrow & \\
 & & & & & \text{Kervaire invariant map.} & 
 \end{array}$$

Thm (Browder) When  $n \neq 2^{\hat{j}+1} - 2$   $\Phi$  is zero.

If  $n = 2^{\hat{j}+1} - 2$   $\Phi$  is non-zero iff

$h_j^2 \in \text{Ext}_{\mathcal{A}}^{2, 2^{\hat{j}+1}}(\mathbb{F}_2, \mathbb{F}_2)$  is a permanent cycle in the  $p=2$  Adams SS of  $\mathbb{S}$ .



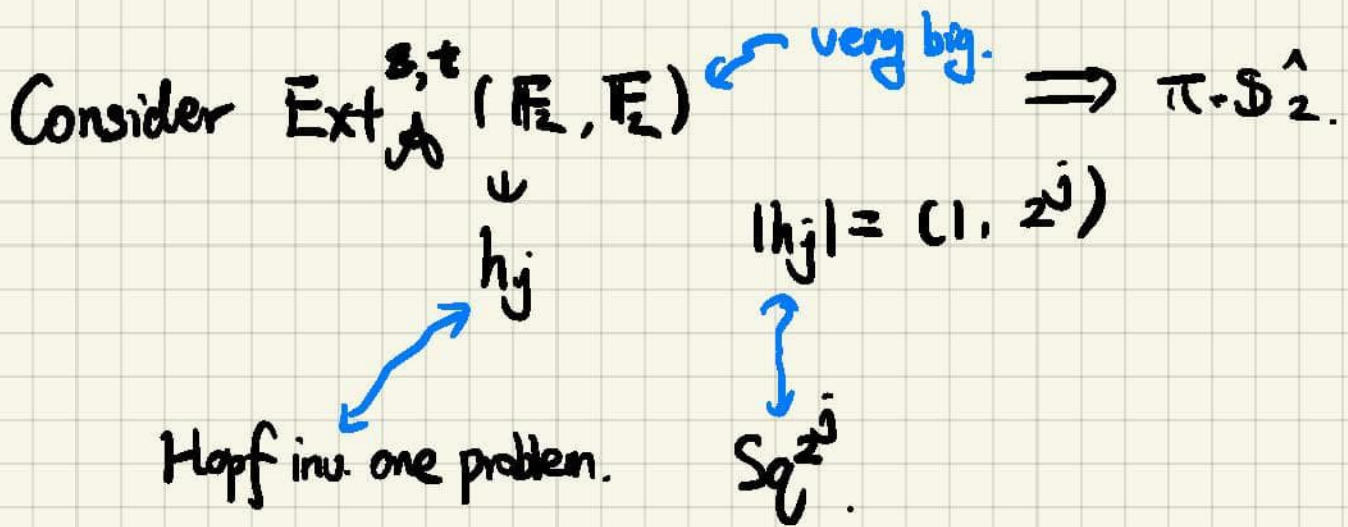
The Adams spectral sequence: Compute  $\pi_* \hat{S}_p$ .

Def. The Steenrod alg.  $\mathcal{A}$  is gen. over  $\mathbb{F}_2$  by

$Sq^i \quad |Sq^i| = i \quad \text{Adem relation:}$   
 If  $i < 2j \quad Sq^i Sq^{2j} = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k.$

\*  $\mathcal{A}$  acts on  $H^*(X; \mathbb{F}_2)$  for all  $X$ :

Thm (Serre)  $\mathcal{A} \cong \pi_* F(H\mathbb{F}_2, H\mathbb{F}_2)$



$\mathcal{A}$  is a Hopf algebra: It has comul. map

$$\Delta(Sq^i) = \sum_{j=0}^i Sq^{i-j} \otimes Sq^j.$$

↓ dual Steenrod algebra

Thm (Milnor) Let  $\mathcal{A}_+ := \text{Hom}_{\mathbb{F}_2}(\mathcal{A}, \mathbb{F}_2)$

then  $\mathcal{A}_+ \cong \mathbb{F}_2[\xi_1, \xi_2, \dots]$   $|\xi_i| = 2^i - 1$

$$\Delta(\xi_i) = \sum_{j=0}^i \xi_{i-j}^{2^j} \otimes \xi_j.$$

UCT:  $\mathcal{A}_+ \cong \pi_* H\mathbb{F}_2 \wedge H\mathbb{F}_2.$

Thm (Adams) There is a SS  $E_2 \cong \text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi \hat{\mathcal{S}}_2$ .

A construction:

$$X_0 = \mathcal{S}$$

$$X_i^! := X_i \wedge H\mathbb{F}_2$$

$$X_{i+1} := \text{hofib}(X_i \rightarrow X_i^!)$$

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ X_2 & \longrightarrow & X_2 \wedge H\mathbb{F}_2 \\ \downarrow & & \\ X_1 & \longrightarrow & X_1 \wedge H\mathbb{F}_2 \\ \downarrow & & \\ \mathcal{S} = X_0 & \longrightarrow & H\mathbb{F}_2 \end{array}$$

Adams SS := the SS of this tower.

Q: Why  $H\mathbb{F}_2$  is special?

Replace  $H\mathbb{F}_2$  by ANY homotopy ring spectrum  $E$ .

The construction works for all  $E$ , but what does it compute?

For nice enough  $E$ , "coalgebra"

$(\pi_* E, \pi_* E \wedge E)$  [Hopf algebroid, affine pre-stack]

The SS has  $E_2 = \text{Ext}_{(\pi_* E, \pi_* E \wedge E)}^{(\pi_* E, \pi_* E)}$

$\Downarrow$   
 $\pi_*$  (some  $E$ -completion of  $\mathcal{S}$ )

Our choice of  $E = MU$ .

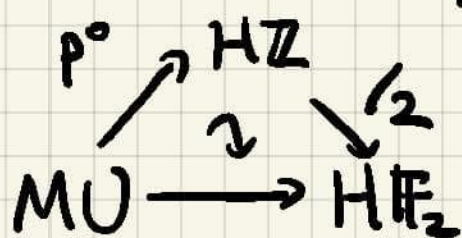


Thm (Quillen)  $(\pi_* MU, \pi_* MU \wedge MU) \hookrightarrow \text{FGL} + \text{strict iso.}$

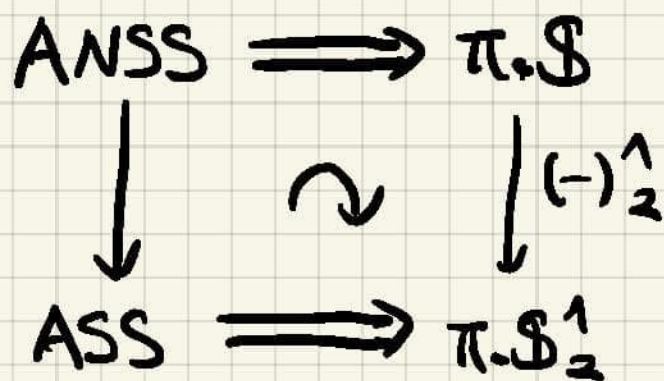
Thm (Novikov) MU-based Adams SS



$\pi_* \mathbb{S}$  without completion.



cplx cobordism  
 & \(\downarrow\)  
 stable htpy gps of spheres.  
 Ravenel.



Another way of thinking  $\text{Ext}(\pi_* MU, \pi_* MU \wedge MU)$

Def.  $\text{MFG}$  is the category of pairs  $(R, F)$

$R$  is a comm. ring  $F$  is a FGL over  $R$ .

$$(f, \psi) : (R_1, F_1) \longrightarrow (R_2, F_2)$$

$$f : R_1 \longrightarrow R_2$$

$$\psi : F_2 \xrightarrow{\cong} f_* F_1$$

Think of  $\mathcal{M}_{FG}$  as a "space" and

$$\text{Ext}_{(\pi, MU, \pi \cdot MU \wedge MU)} \cong H^*(\mathcal{M}_{FG}).$$

Now, given  $(R, F)$  an action of  $G$ :

$g \in G$   $g \curvearrowright R$  by ring automorphism.

choose strict iso.  $g \cdot F \xrightarrow{\cong} F$ .

with compatibility. [  $G$ -object in  $\mathcal{M}_{FG}$  ].

Think of  $G \curvearrowright (R, F)$  as a "subspace" of  $\mathcal{M}_{FG}$ .

$$\text{This gives } H^*(\mathcal{M}_{FG}) \longrightarrow H^*(G; R)$$

$$\text{Ext}_{(\pi, MU, \pi \cdot MU \wedge MU)}$$

"Every time there is a group action on a FGL,  
there is a map from ANSS- $E_2$  to the group cohomology."

— Mike Hopkins.

It can be upgraded:

If  $E$  is a ring spectrum with  $G$ -action. [at least  $A_{00}$ ].

$\pi \cdot E$  has a FGL from  $MU \rightarrow E$

then:



$$\begin{array}{ccc}
 \text{ANSS} & \xRightarrow{\quad} & \pi_* \mathcal{S} \\
 \downarrow & \curvearrowright & \downarrow \\
 H^*(G; \pi_* E) & \xRightarrow{\text{HFPSS}} & \pi_* E^{hG}
 \end{array}$$

Thm (Algebraic detection)

$$\begin{array}{ccc}
 \text{ANSS} & \xrightarrow{\quad} & \text{HFPSS}(\mathcal{D}^{-1} \text{MU}^{\langle G_2 \rangle}) \\
 \downarrow & & \uparrow \pi_* \Omega \\
 \text{ASS} & & 
 \end{array}$$

For all  $x \in \text{Ext}_{\text{MU}, \text{MU}}^{2, 2^{j+1}}(\text{MU}_*, \text{MU}_*) \leftarrow \text{ANSS} - E_2$

s.t.  $x \mapsto h_j^2 \in \text{Ext}_A^{2, 2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$

then  $x \mapsto$  non-zero in  $\text{HFPSS}(\mathcal{D}^{-1} \text{BP}^{\langle G_2 \rangle})$

\* Why  $G_2$ ?

\* How to prove such a thing?



What happens in  $D^{-1}MU^{(G)}$ ?

History: Ravenel asked in 1984:

Does  $v_n^{-1}BP$  split into  $v_n^{-1}BP\langle n \rangle$ ?

Answer: maybe not, but true after  $L_n(-)$

↑  
Hovey - Sudostry.

Our observation:

$\bar{t}_1^{-1}BP_{\mathbb{R}}$ , differentials look like

$\bar{t}_1^{-1}BP_{\mathbb{R}}\langle 1 \rangle$ .

$\bar{t}_n^{-1}BP_{\mathbb{R}}$ , differentials look like

$\bar{t}_n^{-1}BP_{\mathbb{R}}\langle n \rangle$ .

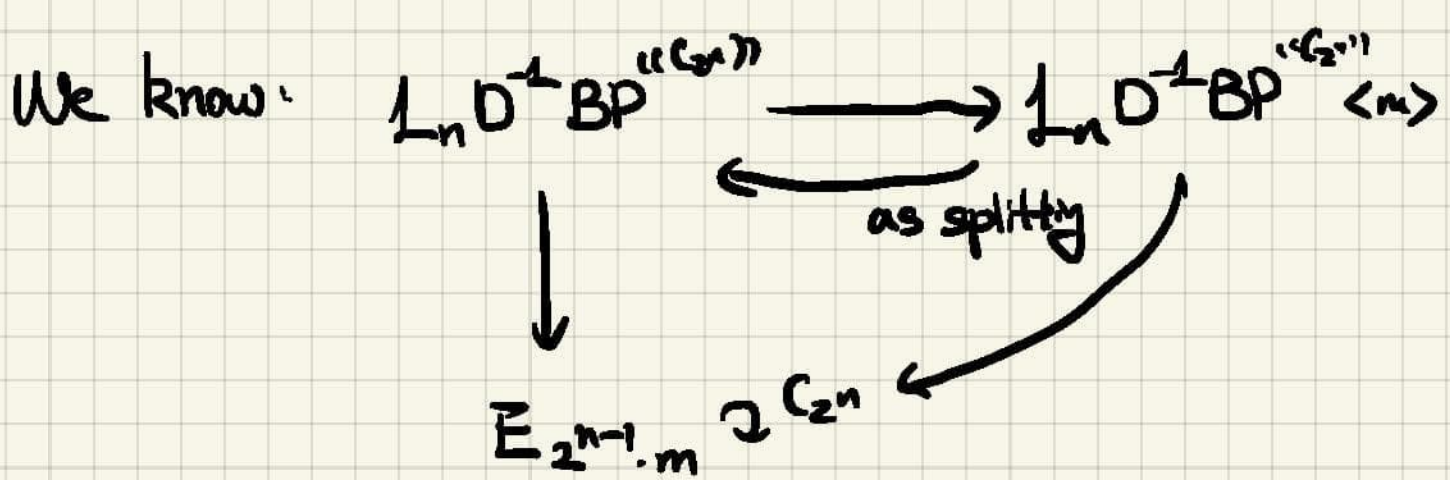
Conjecture 1.  $C_2$ -equivariantly

$L_n \bar{t}_n^{-1}BP_{\mathbb{R}}$  splits into  $L_n \bar{t}_n^{-1}BP_{\mathbb{R}}\langle n \rangle$ .

Conjecture 2.  $C_{2^n}$ -equivariantly

For some smart choice of  $D$ .

$L_n D^{-1}BP^{(C_{2^n})}$  splits into  $L_n D^{-1}BP^{(C_{2^n})}\langle m \rangle$ .



Ref. The non-existence of odd primary Arf invariant elements in stable homotopy. Ravenel 1978.

$E_{\infty}$  - v.s. comm. ring.

Ref. Thm. 1.2.4. of Moduli Problems for Structured Ring Spectra by Goerss-Hopkins.

① Why  $C_8$ ? Automorphism of FGLs.

Def.  $\text{Aut}(F/R)$  is the group of strict auto. of  $F/R$ .

Example.  $F = x+y$   $R = \mathbb{F}_2$ .

$f \in \text{Aut}(F/R)$  if  $f(x) + f(y) = f(x+y)$

$$f(x) = \sum a_i x^{i+1} \implies \sum a_i x^{i+1} + \sum a_j y^{j+1} = \sum a_i (x+y)^{i+1}$$

$$\implies a_i = 0 \text{ unless } i = 2^k - 1.$$



$$\begin{array}{l}
 \mathbb{Z}[G] \\
 H^i(G; M) := \text{Ext}_{\mathbb{Z}[G]}^i(\mathbb{Z}, M) \\
 \mathbb{Z}[G]^\times \\
 H^i(G; M) := \text{Ext}_{\mathbb{Z}[G]^\times}^i(\mathbb{Z}, M)
 \end{array}$$



$$f(x) = \sum \xi_i x^{2^i}$$

Given any  $R$   $\mathbb{F}_2$ -alg.  $\text{Aut}(x^{\pm 1}/R)$   
 $\mathbb{F}_2\text{-alg}(\mathcal{A}_+, R) \cong$

$$\text{Aut}(F/R) \cong \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

$\begin{matrix} \text{SII} \\ \mathcal{A}_+ \end{matrix}$

$|\xi_i| = 2^i - 1$   
 if  $|x| = -1$   
 $|f(x)| = -1.$

$\Delta(\xi_i) =$  composition of power series.

$$= \sum_{j=0}^i \xi_{i-j}^{2^j} \circ \xi_j.$$

$$\text{Adams SS-E}_2 \cong H^*(\text{Aut}(F/R); \mathbb{F}_2).$$

Def. a FGL  $F/R$   $R$ : comm.  $\mathbb{F}_p$ -alg.

$F$  has height  $h$  if

$$[p]_F(x) = ax^{p^h} + \text{higher terms.}$$

if  $[p]_F(x) = 0$  height =  $\infty$ .

- height is iso. inv.
- height classifies FGLs / separably closed fields of char.  $p$ .



Thm (Hewett) Let  $F_h$  be a FGL of height  $h$ .  
over  $\mathbb{F}_2$ , then

$$C_{2^n} \subseteq \text{Aut}(F_h) \text{ iff } h = 2^{n-1} \cdot m \\ \text{for } m > 0.$$

Reminder:  $BP_* \cong \mathbb{Z}_{(2)}[t_1^{c_2}, t_2^{c_2}, \dots]$

$t_i^{c_2}$  represents height  $i$ .

Thm (Lubin-Tate) [universal deformation]

There is a complete local ring  $E_{h,*}$ , and a FGL  $\hat{F}_h$  over it.

s.t. ①  $E_{h,*}/\mathfrak{m} \cong \mathbb{F}_2$  and  $\hat{F}_h/\mathfrak{m}$  has height  $h$ .

②  $\hat{F}_h$  is  $p$ -typical and  $t_1^{c_2}, \dots, t_h^{c_2}$  form  
a regular sequence in  $E_{h,*}$ .

$(E_{h,*}, \hat{F}_h)$



is like "universal cover of spaces".

$(\mathbb{F}_2, \hat{F}_h/\mathfrak{m})$

$$\text{Aut}(\hat{F}_h/\mathfrak{m}, \mathbb{F}_2) \curvearrowright (E_{h,*}, \hat{F}_h)$$

Thm (Goerss - Hopkins - Miller)

There is comm. ring spectrum  $E_h$  s.t.

•  $\pi_* E_h \cong E_{h-1}$  with  $FG \perp \tilde{F}_h$ .

•  $\text{Aut}(\tilde{F}_h/m) \simeq E_h$  as comm. ring maps.

$h=4$   $C_8 \simeq E_4$ .

Thm (Ravenel, 1978)  $C_p \simeq E_{p-1}$  for  $p$  odd.

$p > 3$ .  $E_{p-1}^{hC_p}$  detects odd primary analog of the Kervaire inv. one elements.

HHR v1.  $C_8 \simeq E_4$ .  $E_4^{hC_8}$  detects all Kervaire inv. one elements.

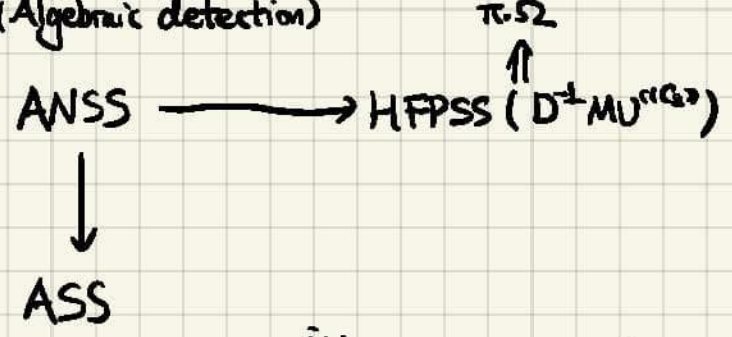
Problem:  $E_4^{hC_8}$  is difficult to compute.

open as of today.

Turn to  $MU^{C_8} \longrightarrow E_4$   $C_8$ -equivariant.

(2) How to prove the detection thm?

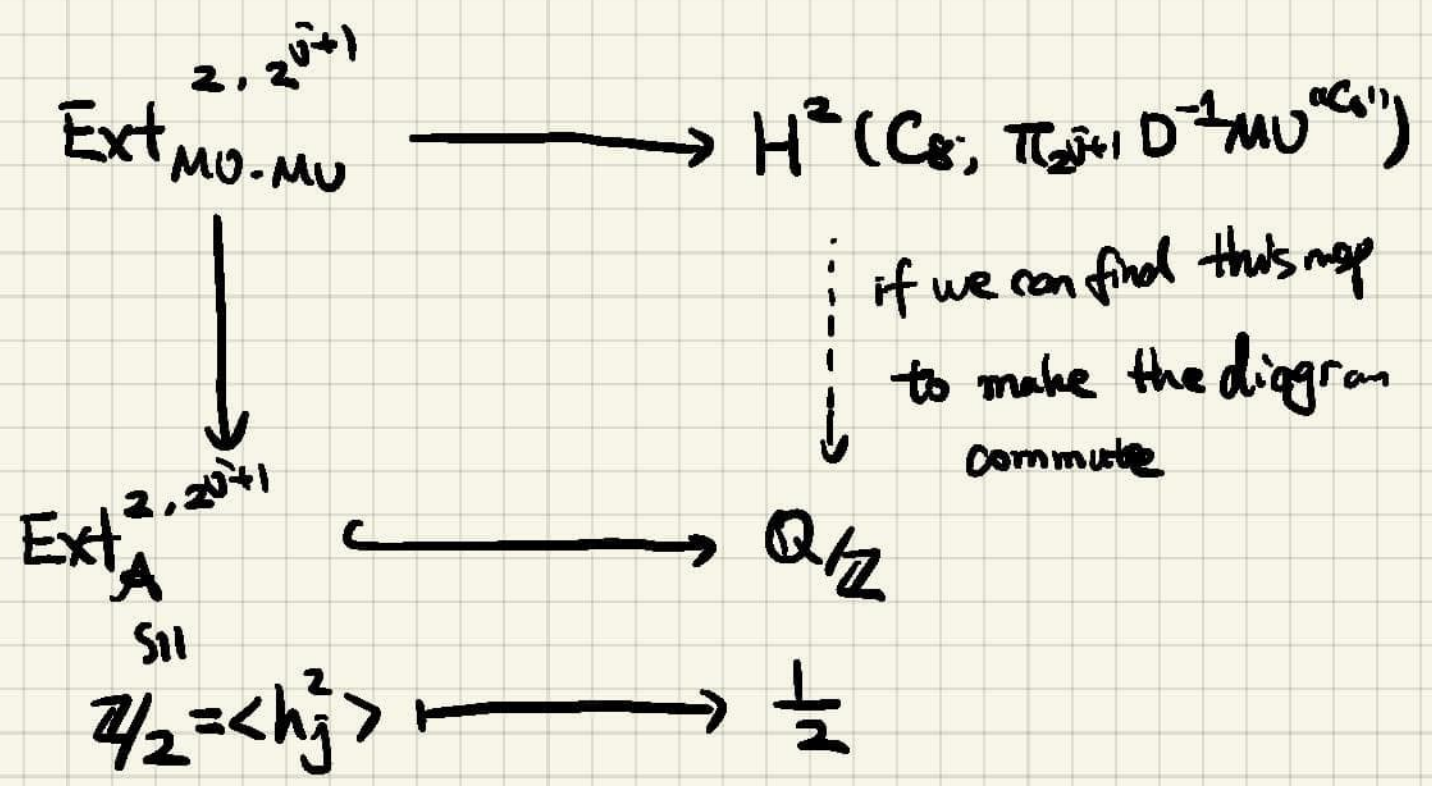
Thm (Algebraic detection)



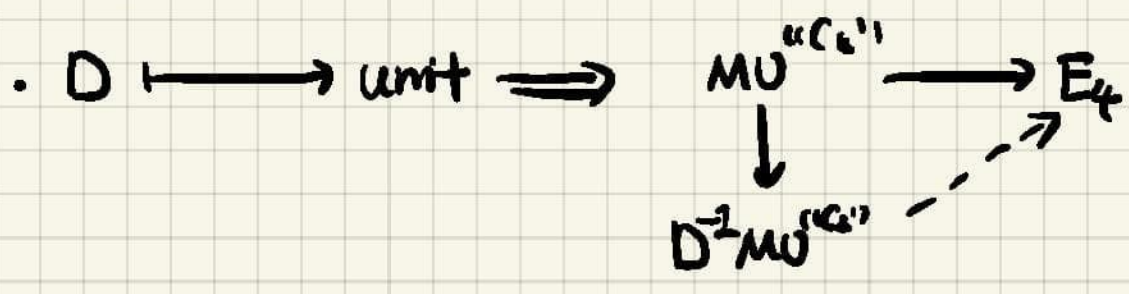
For all  $\pi \in \text{Ext}_{MU, MU}^{2, 2^{j+1}}(MU_*, MU_*) \leftarrow \text{ANSS} - E_2$

s.t.  $\pi \mapsto h_j^2 \in \text{Ext}_A^{2, 2^{j+1}}(E, E)$

then  $\pi \mapsto$  non-zero in  $\text{HFPSS}(D^{-1}BP^{(G)})$



Original: map into  $H^2(C_8; E_{4r})$



• HHR version 1.



v4: maps  $H^2(C_8; \pi_{2^{\bar{v}+1}} D^{-1} MU^{(C_8)})$  into a smaller ring.

$A := \mathbb{Z}_2[\xi]$   $\xi$  is a 8th root of 1.

$A_* := A[u^{\pm}]$   $|u| = 2$

$C_8 \simeq A_*$  via  $u \mapsto \xi u$ .

Lubin-Tate (formal  $A$ -module)

$\exists FGL F/A_*$

• height  $(F/m) = 4$  we need  $a=f$

•  $C_8 \simeq (A_*, F)$ .

[ $A_*$  is NOT  $\pi_*$  of something] - Andrew Sutch

This gives:  $\pi_*^u MU^{(C_8)} \longrightarrow A_*$   $C_8$ -ring map.

- $i_e^*(D) \longmapsto$  unit in  $A_*$ .  
where the choice of  $D$  matters.

$\chi: H^2(C_8; \pi_{2^{\bar{v}+1}} D^{-1} MU^{(C_8)}) \longrightarrow H^2(C_8; \mathbb{A}_{2^{\bar{v}+1}})$

$\pi = \xi - 1$  uniformizer.

diagram:

F FGL.  
 $\text{Int}_F: n \in \mathbb{Z}$ .  
 extend:  $n \in \mathbb{Z}_p$   
 $\mathbb{Z}_p \hookrightarrow A$   
 Hope to define  
 $\forall a \in A$   
 $\text{Int}_F$  st. it generates  
 $\text{Int}_F: n \in \mathbb{Z}_p$ .

$$\text{Ext}_{\text{MU-MU}}^{2, 2^{\bar{j}+1}} \longrightarrow H^2(\mathbb{C}_8; \pi_{2^{\bar{j}+1}}^{\text{MU}} D^{-1} \text{MU}^{\langle \mathbb{C}_8 \rangle})$$

$$\downarrow$$

$$\text{Ext}_A^{2, 2^{\bar{j}+1}} \cong \mathbb{Z}/2$$

SII

(?)

$$\downarrow \frac{2}{\pi} \cdot \chi$$

$$H^2(\mathbb{C}_8; A_{2^{\bar{j}+1}})$$

4x generator.

SII  $\sim j > 3$   $\mathbb{C}_8 \simeq A_{2^{\bar{j}+1}}$  trivially.

$$H^1(\mathbb{C}_8; A_{2^{\bar{j}+1}} / \langle \pi \rangle) \cong \mathbb{Z}/2$$

$$\begin{array}{ccc} & \nearrow A_{2^{\bar{j}+1}} / \langle \pi \rangle & \\ & \longrightarrow \mathbb{Q}/\mathbb{Z} & \xrightarrow{\text{gen}} \frac{1}{8} \\ & \longmapsto \frac{1}{2} & \end{array}$$

If the diagram commutes we are done.

Hard: • Understand  $\text{Ext}_{\text{MU-MU}}^{2, 2^{\bar{j}+1}}$  [Shimomura]

$$\text{Ext}_{\text{MU-MU}}^{2, 2^{\bar{j}+1}} \longrightarrow \text{Ext}_A^{2, 2^{\bar{j}+1}}$$

and

$$\text{Ext}_{\text{MU-MU}}^{2, 2^{\bar{j}+1}} \longrightarrow H^2(\mathbb{C}_8; A_{2^{\bar{j}+1}})$$

↳ determined by the FGL  $\tilde{F}$  on  $\mathbb{A}_*$ .

check commutativity. Formal  $A$ -modules.  $\square$

Why HFPSS is hard?

By GHM:  $C_8 \sim E_{4+} = \pi_{S11} E_4.$

$C_8 \sim \Psi(\mathbb{F}_2) \Pi[\mu_1, \mu_2, \mu_3] [U^*]$

Difficulty 1: How  $G \sim \pi_{E_h} \bar{E}_h$  is not clear.

Difficulty 2: -Differentials.