Cohomology theories, Brown representability & K-theory.

<u>Defn</u>. the homotopy category of pointed spaces is a category $h\mathcal{S}_*$:

obj: pointed CW complexes $* \to X$.

mor: $\mathrm{Hom}_{h\mathcal{S}_*}(X, Y) = \pi_0 \mathrm{Map}_*(X, Y)$.

composition: take the composition map then apply π_0.

let $h\mathcal{S}_*^{\geq 0}$ denote the full subcat whose objects are pointed connected CW complexes.

<u>Defn</u>. a generalised homology theory is a sequence of functors $\{H_n : h\mathcal{S}_* \to \mathcal{A}b\}$ ~~together~~ with natural isomorphisms $\{\partial_n : H_n \circ \Sigma \to H_{n-1}\}$ satisfying

1) (exactness) for any cofibre sequence $A \to X \to X/A$, the sequence $H_n(A) \to H_n(X) \to H_n(X/A)$ is exact

2) H_n preserves coproducts, i.e.

$$\bigoplus_{i\in I} H_n(X_i) \to H_n(\bigvee_{i\in I} X_i) \text{ is an isomorphism}$$

Similarly, a generalized cohomology ~~theory~~ is a sequence of functors $\{H^n : (h\mathcal{T}_*)^{op} \to \mathcal{Ab}\}$ together with natural isomorphisms $\{\delta^n : H^n \to H^{n+1}\circ\Sigma\}$, satisfying

1) (exactness) for any cofibre sequence $A \to X \to X/A$, the sequence $H^n(X/A) \to H^n(X) \to H^n(A)$ is exact.

2) H^n takes coproducts to products, i.e.

$$H^n(\bigvee_{i\in I} X_i) \to \prod_{i\in I} H^n(X_i) \text{ is an isomorphism}$$

NOTE: we can define the "boundary homomorphisms"

$$\partial : H^n(A) \to H^{n+1}(\Sigma A) \to H^{n+1}(X/A)$$

and there's a LES by the Puppe coexact sequence

$$\cdots \to H^{n-1}(A) \to H^n(X/A) \to H^n(X) \to H^n(A) \to \cdots$$

Brown representability theorem:

$F : (h\mathcal{S}_*^{\geq 0})^{op} \to \mathcal{S}et$ is representable iff

1) it takes coproducts to products.

2) for pointed CW triple $(X; A, B)$ with $X = A \cup B$,

$F(X) \to F(A) \times_{F(A\cap B)} F(B)$ is surjective

<u>Cor.</u> for a generalized cohomology theory $\{H^n, \delta^n\}$,

every H^n is representable.

pf: note that $H^n(X) \cong H^{n+1}(\Sigma X)$. if $H^{n+1} : (h\mathcal{S}_*^{\geq 0})^{op} \to \mathcal{A}b$

is represented by $Y_{n+1} \in h\mathcal{S}_*^{\geq 0}$. then

$H^n(X) \cong \mathrm{Hom}_{h\mathcal{S}_*^{\geq 0}}(\Sigma X, Y_{n+1}) \cong \mathrm{Hom}_{h\mathcal{S}_*}(X, \Omega Y_{n+1})$

is represented by $\Omega Y_{n+1} \in h\mathcal{S}_*$

then the condition 2) in Brown rep. thm follows from

$$\begin{tikzcd}
H^n(X/B) \arrow[r] \arrow[d, "\cong"'] & H^n(X) \arrow[r] \arrow[d] & H^n(B) \arrow[r] \arrow[d] & H^{n+1}(X/B) \arrow[d, "\cong"'] \\
H^n(A/A\cap B) \arrow[r] & H^n(A) \arrow[r] & H^n(A\cap B) \arrow[r] & H^{n+1}(A/A\cap B)
\end{tikzcd}$$

by diagram chasing $\square$

then for any $\{H^n, \delta^n\}$, $\exists\, Y_n \in h\mathcal{S}_*$, satisfying

$$H^n(X) \cong \mathrm{Hom}_{h\mathcal{S}_*}(X, Y_n) \cong \pi_0 \mathrm{Map}_*(X, Y_n)$$

and $\delta^n : H^n \xrightarrow{\cong} H^{n+1} \circ \Sigma$ induces

$$\mathrm{Hom}_{h\mathcal{S}_*}(X, Y_n) \xrightarrow{\cong} \mathrm{Hom}_{h\mathcal{S}_*}(\Sigma X, Y_{n+1}) \cong \mathrm{Hom}_{h\mathcal{S}_*}(X, \Omega Y_{n+1})$$

$\Rightarrow Y_n \to \Omega Y_{n+1}$ is an isomorphism in $h\mathcal{S}_*$,

hence a weak homotopy equivalence

<u>Defn</u>. an Ω-spectrum is a sequence $K_0, K_1 \cdots \in h\mathcal{S}_*$, together with weak homotopy equivalences $\{K_n \to \Omega K_{n+1}\}$.

NOTE: we can set $K_{-n} = \Omega^n K_0$ for $n > 0$.

as such, every cohomology theory is represented by an Ω-spectrum.

e.g. for $A \in \mathcal{A}b$, $\tilde{H}^n(X; A) \cong \mathrm{Hom}_{h\mathcal{S}_*}(X, K(A,n))$ is represented by the Eilenberg-MacLane Ω-spectrum $(HA)_n = K(A,n)$

e.g. K-theory

for $X \in h\mathcal{S}_*$, let $\mathrm{Vect}_k(X)/\cong$ be the set of isomorphism classes of k-vector bundles over X ($k = \mathbb{R}, \mathbb{C}$).

the direct sum $\oplus$ makes it a commutative monoid

take the group completion $K(\mathrm{Vect}_k(X)/\cong, \oplus) = \{E - E'\}/\sim$

if X is compact, then $\forall E$, $\exists E'$, s.t. $E \oplus E' \cong \varepsilon^n$.

then elements in $K(\mathrm{Vect}_k(X)/\cong, \oplus)$ has the form $E - \varepsilon^n$.

for the case $k = \mathbb{C}$, $K(\mathrm{Vect}_k(X)/\cong, \oplus) \cong [X, BU \times \mathbb{Z}]$,

where $BU = \varinjlim BU(n)$; $\mathbb{Z}$ classifies the rank.

esp. $K(\mathrm{Vect}_k(*)/\cong, \oplus) \cong [*, BU \times \mathbb{Z}] \cong \mathbb{Z}$.

let $KU^0(X) = \ker([X, BU \times \mathbb{Z}] \to [*, BU \times \mathbb{Z}])$

$$\cong \mathrm{Hom}_{h\mathcal{S}_*}(X, BU \times \mathbb{Z}).$$

the complex K-theory Ω-spectrum KU:

$(KU)_0 = BU \times \mathbb{Z} \rightsquigarrow (KU)_{-1} = \Omega (KU)_0 \simeq U$,

$$(KU)_{-2} = \Omega (KU)_{-1} \simeq BU \times \mathbb{Z}$$

define $(KU)_{2n} = BU \times \mathbb{Z}$. $(KU)_{2n+1} = U$.

similarly, for the case $k=\mathbb{R}$, there's the real K-theory Ω-spectrum KO:

$(KO)_0 = BO \times \mathbb{Z}$. $\rightsquigarrow \Omega^8(BO\times\mathbb{Z}) \simeq BO \times \mathbb{Z}$

define $(KO)_{8n} = BO \times \mathbb{Z}$. $(KO)_{8n-k} = \Omega^k(BO\times\mathbb{Z})$.