Complex bordism, Thom construction

- $\pi_n(X)$ is not a homology theory;

  $\pi_n^s(X) = \varinjlim \pi_{n+k}(\Sigma^k X)$ is a homology theory.

<u>Defn</u>. a <u>spectrum</u> K consists of a sequence $K_0, K_1 \cdots \in h\mathscr{T}_*$, ~~together~~ with structure maps $\Sigma K_n \to K_{n+1}$

the homotopy groups of K : $\pi_n K = \varinjlim \pi_{n+i} K_i$

e.g. for $X \in h\mathscr{T}_*$, the suspension spectrum $(\Sigma^\infty X)_n = \Sigma^n X$.

its homotopy groups are $\pi_n(\Sigma^\infty X) = \pi_n^s X$.

e.g. for an Ω-spectrum K, its associated spectrum $(K')_n = K_n$; $\Sigma K_n \to K_{n+1}$ is the adjoint of $K_n \longrightarrow \Omega K_{n+1}$

e.g. K is a spectrum, $X \in h\mathscr{T}_*$. we can construct a spectrum $(K \wedge X)_n = K_n \wedge X$.

in particular, $\Sigma^\infty X = (\Sigma^\infty S^0) \wedge X$,
and $\pi_n^s X = \pi_n\left((\Sigma^\infty S^0) \wedge X\right)$.

more generally, it can be shown that for any spectrum K, $\pi_n(K \wedge -)$ is a homology theory.

(Adams) any homology theory satisfying $H_n(X) \cong \varinjlim H_n(X_\alpha)$ (where X_α takes over all finite subcomplexes of X) can be defined by a spectrum K, i.e. $H_n(X) \cong \pi_n(K \wedge X)$.

e.g. for $A \in \mathcal{Ab}$, $\tilde{H}_n(-;A)$ is defined by the Eilenberg-MacLane spectrum HA, the associated spectrum of Eilenberg-MacLane Ω-spectrum. i.e.
$(HA)_n = K(A,n)$, $\Sigma K(A,n) \to K(A,n+1)$: the adjoint maps

e.g. the bordism homology theories are defined by some Thom spectra:
(Reference: Denis Nardin's lecture notes on his homepage)

give a map of pointed spaces $f: X \to BO$, we can construct a Thom spectrum X^f as a generalization of Thom spaces.

- if f factors through some $BO(n)$, and classifies a virtual vector bundle of rank 0: $\gamma - \varepsilon^n$, then we can set $(X^f)_k = \begin{cases} * & , k < n \\ \Sigma^{k-n} Th(\gamma), & k \geq n. \end{cases}$

- when f is the identity map $BO \to BO$, the Thom spectrum is called the bordism spectrum MO: $(MO)_n = Th(\gamma_n)$: γ_n is the universal n-real bundle.

  structure maps $\Sigma Th(\gamma_n) \cong Th(\gamma_n \oplus \varepsilon) \to Th(\gamma_{n+1})$ come from the maps of bundles

$$\begin{tikzcd} \gamma_n \oplus \varepsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & \gamma_{n+1} \arrow[d] \\ BO(n) \arrow[r] & BO(n+1) \end{tikzcd}$$

- when f is the map $BSO \to BO$, the Thom spectrum is called the oriented bordism spectrum MSO: $(MSO)_n = Th(\xi_n)$: ξ_n is the universal oriented bundle.

- when f is the map $BU \to BO$ induced by $U(n) \hookrightarrow O(2n)$ the Thom spectrum is called the complex bordism spectrum:
  $(MU)_{2n} = Th(\gamma_n^{\mathbb{C}})$; $\gamma_n^{\mathbb{C}}$ is the universal n-complex bundle.
  $(MU)_{2n+1} = \Sigma Th(\gamma_n^{\mathbb{C}})$;
  the structure maps come from
$$\begin{tikzcd} \gamma_n^{\mathbb{C}} \oplus \varepsilon^{\mathbb{C}} \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & \gamma_{n+1}^{\mathbb{C}} \arrow[d] \\ BU(n) \arrow[r] & BU(n+1) \end{tikzcd}$$

- when f is the map $PBO \to BO$, where PBO is the path space, the Thom spectrum is
  $(X^f)_n = Th(PBO(n) \to BO(n)) \simeq S^n$ since $PBO(n) \simeq *$.

Now fix a fibration $\xi : B \to BO$

<u>Defn</u>: given a compact space X and a map $V : X \to BO$, a <u>ξ-structure</u> on V is a factorization
$$\begin{tikzcd} & B \arrow[d, "\xi"] \\ X \arrow[ur, dashed, "V_\xi"] \arrow[r, "V"] & BO \end{tikzcd}$$

let $M\xi$ be the Thom spectrum of ξ

- when $B = BSO$, ξ-structure is an orientation.
- when $B = BU$, ξ-structure is stable complex structure
- when $B = PBO$, ξ-structure is stable framing.

<u>Defn</u>: let M be a compact n-manifold, a ξ-structure on M is a ξ-structure on its stable normal bundle $\varepsilon^n - TM$.

NOTE: for a $(n+1)$-manifold M with boundary, $(\varepsilon^{n+1} - TM)|_{\partial M} \cong \varepsilon^n - T\partial M$. thus, a ξ-structure on M will induce the ξ-structure on ∂M:

$$
\begin{tikzcd}
 & & B \arrow[d, "\xi"] \\
\partial M \arrow[r] \arrow[urr, dashed] & M \arrow[ur] \arrow[r, "\varepsilon^{n+1} - TM"] & BO
\end{tikzcd}
$$

<u>Defn</u>: for $n \geq 0$, the <u>n-dimensional cobordism group</u> is

$$
\Omega_n^\xi = \frac{(\{\text{closed } n\text{-manifolds with } \xi\text{-structure}\}, \sqcup)}{\langle \text{boundaries of } (n+1)\text{-manifolds with } \xi\text{-structure} \rangle}
$$

- Ω_n^{ξ} is a group:

for
$$\begin{tikzcd} & B \arrow[d] \\ M \arrow[r] \arrow[ur, "\varphi"] & BO \end{tikzcd}$$
, we can consider the lifting problem

$$\begin{tikzcd} M\times\{0\} \arrow[r, "\varphi"] \arrow[d] & B \arrow[d, "\xi"] \\ M\times I \arrow[r] \arrow[ur, dashed, "\tilde{\varphi}"] & BO \end{tikzcd}$$
, which has a solution since ξ is a fibration. then its inverse is given by

$$\begin{tikzcd} & & B \arrow[d] \\ M\times\{1\} \arrow[r] \arrow[urr, dashed, "\varphi'"] & M\times I \arrow[r] \arrow[ur, "\tilde{\varphi}"] & BO \end{tikzcd}$$

Thom construction: for $(M, \varphi) \in \Omega_n^{\xi}$, by Whitney's embedding theorem, M can be embedded in some $\mathbb{R}^{n+k}$, hence S^{n+k}. then we have the map:

$$S^{n+k} \longrightarrow S^{n+k}/(S^{n+k} - tb(M)) \cong Th(\varepsilon^{n+k} - TM)$$

this gives an element in $\pi_{n+k} Th(\varepsilon^{n+k} - TM) \cong \pi_{n+k}(M^{\varepsilon^n - TM})_k$

take the colimit with respect to k, we get an element in $\pi_n(M^{\varepsilon^n - TM})$. so we get a map

$$\Omega_n^{\xi} \longrightarrow \pi_n(M^{\varepsilon^n - TM}) \xrightarrow{\varphi_*} \pi_n M\xi$$

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\varepsilon^n - TM"'] & B \arrow[d, "\xi"] \\ & BO \end{tikzcd}$$

Pontryagin - Thom theorem: $\Omega_n^{\xi} \xrightarrow{\cong} \pi_n M\xi$

esp. for $\xi : PBO \to BO$, $\pi_n M\xi \cong \varinjlim \pi_{n+k} S^k \cong \pi_n^s S^0$.

note that $M\xi \wedge (X_+) \simeq M\xi_X$, where $\xi_X : B \times X \to B \to BO$

the unreduced bordism homology theory

$\pi_n(M\xi \wedge (X_+)) \cong \Omega_n^{\xi_X}$ can be described as equivalent classes

of (M, φ, f) : φ is a ξ-structure on M, $f : M \to X$.