1. motivation

Spanier - Whitehead category: obj: pointed CW cplx
mor: $\mathrm{Hom}(X,Y) := \varinjlim_{q} [\Sigma^q X, \Sigma^q Y]$

made possible by the **Freudenthal suspension theorem**.

more generally, we can consider the part with negative homotopy (non-connective)
obj: pairs (X,n) X pointed CW cplx $n \in \mathbb{Z}$.
mor: $\mathrm{Hom}((X,n),(Y,m)) := \varinjlim_{q} [\Sigma^{q+n} X, \Sigma^{q+m} Y]$.

This is a triangulated category. But it doesn't satisfy some prop.

does not preserve coprod $\rightsquigarrow$ Brown representability not hold.

2. basic notions.

a Spectrum has the following data: $E_n \in Top_*$, $n \in \mathbb{Z}$.
structure maps: $\sigma_n : \Sigma E_n \to E_{n+1}$.

by putting various of conditions on E_n and σ_n, we can obtain special spectra.

- $\Sigma E_n \to E_{n+1}$ is identity $\rightsquigarrow$ suspension spec. e.g. $\mathbb{S}$ sphere spectrum.
- E_n are CW-cplx, σ_n are inclusions of subcplx. $\rightsquigarrow$ CW-spec.
- the map $E_n \to \Omega E_{n+1}$ corresponding to σ_n is an iso for every n. $\rightsquigarrow$ Ω-spec.

from previous talk, spectra represent cohomology theories:

Let $\tilde{E}^*(-)$ be a reduced cohomology theory, then $\exists$ an Ω-spec s.t.
$\tilde{E}^n(X) \cong [X, E_n]$ for each n.

- $\tilde{E}^*(-) = \tilde{H}^*(-;A) \quad \longleftrightarrow \quad E = HA$ Eilenberg McLane spectrum
- $KU \qquad KU$
- $MU \qquad MU$

spectra has good homotopy category structure. And it has strong connection with cohomology theories. So we may expect the structures/properties of cohom theories to show up in spectra.

3. product structure.

- $\tilde{H}^*(-)$ cup prod.
- $KU^*(-)$ (Grothendieck construction of v.b). tensor of v.b. induced.
- $MU^*(-)$ Cartesian prod of manifolds.

Künneth theorem: in good situation identifies the cohomology of a product with the tensor prod of cohomologies.

$$\tilde{E}^n(X) \otimes \tilde{E}^m(Y) \to \tilde{E}^{m+n}(X \wedge Y)$$

because of the representibility, cohomologies are almost a "mapping space".

$\tilde{E}^*(X) \cong [X, E]_*$. the multiplicative structure on this relies heavily on the target spaces.

motivating illustration:

$$[X, E_n] \otimes [X, E_m] \to [X \wedge X, E_n \wedge E_m] \xrightarrow{\Delta} [X, E_n \wedge E_m] \longrightarrow [X, E_{n+m}]$$

we want a product on spectra!

4. smash product.

structure map hard to define where to go. $\Sigma(X \wedge Y)_n \to (X \wedge Y)_{n+1}$

$$\begin{array}{ccc} & \cdots\; X_n & X_{n+1} \;\cdots \\ \vdots & & \\ Y_n & \bullet \xrightarrow{?} & \\ Y_{n+1} & ?\downarrow & \\ \vdots & & \end{array}$$

(well explained in Adams).

to make a good cat of spectra

But we just want a product! what is considered a "qualified" (good) one?

A1 The category Sp is a symmetric monoidal cat with respect to the smash product.

A2. There exists a lax monoidal adjunction

$$\Sigma^\infty : Top_* \rightleftarrows Sp : \Omega^\infty$$

implies

A3. The unit for the smash product in Sp is the sphere spectrum (The map $\Sigma^\infty S^0 \to$ Unit is an iso).

A4. Either $\exists$ a natural transformation

$$\phi : (\Omega^\infty D) \wedge (\Omega^\infty E) \to \Omega^\infty(D \wedge E)$$

or $\exists$ a natural transformation

$$\gamma : \Sigma^\infty(X \wedge Y) \to (\Sigma^\infty X) \wedge (\Sigma^\infty Y)$$

A5. let Q be the stabilization functor $QX := \operatorname{colim}_n \Omega^n \Sigma^n X$

$\exists$ a natural weak homotopy equivalence f.

s.t.

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\iota"'] & \Omega^\infty \Sigma^\infty X \arrow[d, "f"] \\ & QX \end{tikzcd}$$

commute.

Lewis'
Thm (1990)
There's no notion of cat of spectra satisfies $A_1 \sim A_5$.

for those can't read Chinese, this says "but I can't make it" in a funny way :)

sketch proof:

Since S is the unit, it is a commutative monoid in Sp.
Ω^∞ is a lax Symm moidal functor, so $\Omega^\infty S$ is also a commutative monoid.
Therefore QS^0 is a commutative monoid.

However, Moore's thm says. there's not that many things have strict ring structure.

If Sp is a cat satisfies $A1 \sim A4$, E is a strict ring spectrum,
then $\Omega^\infty E \simeq \prod EM$.

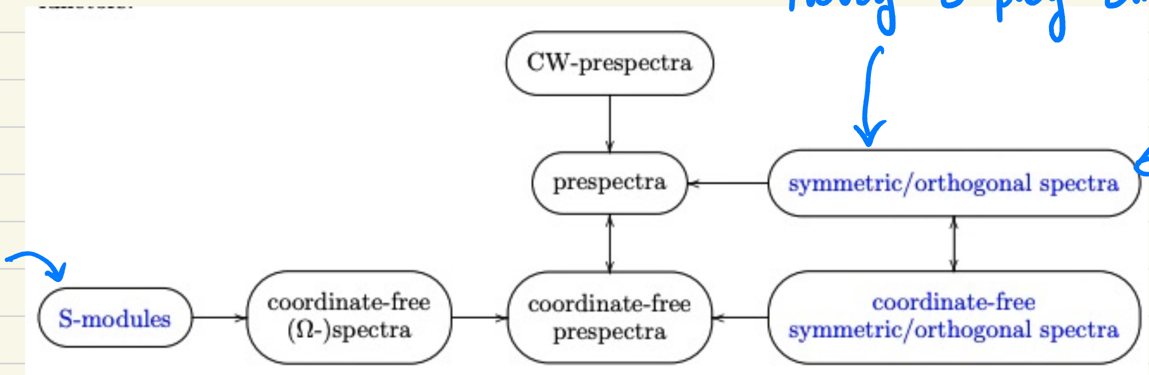
So. $QS^0 \simeq \prod EM$ which is false.

5. different models.

Hovey - Shipley - Smith 2000

Mandell - May - Schwede - Shipley 1998

Elmendorf - Kriz - Mandell - May 1997



- EKMM Spectra

take use of the linear isometry operad to make coordinate free.

Fix a universe $\mathcal{U} \cong \mathbb{R}^\infty$, a prespectrum E has data:

- $E(V) \in Top_*$ for fin dim inner product vector space V.
- $\sigma_{V,W} : \Sigma^{W-V} E(V) \to E(W)$ for $V \subseteq W$.

if $\sigma_{V,W}$ is a homeomorphism for $\forall\, V \subseteq W \subseteq \mathcal{U}$, E is called a Lewis-May-Steinberger spectrum. $\alpha : \mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$ sending $(V, V') \mapsto W$

How to construct the smash product? $E \wedge_\alpha F(W) = E \wedge F(V, V') = E(V) \wedge F(V')$

don't want to depend on α. the linear isometry operad $\mathcal{L}(n) := \mathcal{L}(\mathcal{U}^{\oplus n}, \mathcal{U})$

notice that, $\mathcal{L}(2)$ = the space of all choices $\mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$ is contractible.

So they construct a twisted half smash product

$\mathcal{L}(2) \ltimes (E \wedge F)$ and fix the associativity by quotienting out $\mathcal{L}(1) \times \mathcal{L}(1)$.

$$E \wedge_{\mathcal{L}} F = \mathcal{L}(2) \ltimes_{\mathcal{L}(1) \times \mathcal{L}(1)} (E \wedge F).$$

then make $\wedge_{\mathcal{L}}$ unital by restricting to all E with $E \wedge_{\mathcal{L}} S \cong E$. (S-modules)

- Symmetric spectra.

E consists of $E_n \in Top_*$ for $n \in \mathbb{N}$.

structure maps $\sigma_n : \Sigma E_n \to E_{n+1}$

Σ_n-action on E_n s.t. the composite.

$$S^p \wedge E_q \to S^{p-1} \wedge E_{1+q} \to \cdots \to S^1 \wedge E_{p-1+q} \to E_{p+q} \quad \text{is } (\Sigma_p \times \Sigma_q)\text{-equivariant.}$$

Smash prod: $$(E \wedge F)_n = \bigvee_{p+q=n} \Sigma_{p+q\,+} \wedge_{\Sigma_p \times \Sigma_q} (E_p \wedge F_q) / \sim .$$

where the $\sim$ identifies.

$$\begin{array}{ccc} & \nearrow & \Sigma_{p+q+r\,+} \wedge_{\Sigma_q \times \Sigma_{p+r}} (E_q \wedge F_{p+r}) \\ & & (\sigma \circ \tau_{qp}, x, sy) \\ \Sigma_{p+q+r\,+} \wedge (S^p \wedge E_q \wedge F_r) & & \\ (\sigma, s, x, y) & & \\ & \searrow & \Sigma_{p+q+r\,+} \wedge_{\Sigma_{p+q} \times \Sigma_r} (E_{p+q} \wedge F_r) \\ & & (\sigma, sx, y) \end{array}$$

- Orthogonal Spectra. replace Σ_n action with $O(n)$ action

- Pro and cons.

EKMM, Sp^{Σ}, Sp^{O} all fail A5.

Sp^{O}: easy to equivariantize.
all obj. fibrant.
Stable π_* is w.e.

Sp^{Σ}: $\exists$ convenient model str. on comm ring.
make a lot more sense if use sset.
stab π_* is not w.e.

EKMM: Ω^∞ give 0-th space information
all obj. fibrant.
stable π_* is w.e.
hard to define smash prod.

- Stable homotopy category.

S-modules. Sp^{Σ} and Sp^{O} all have good smash products before passing to the homotopy cat.

Sp^{Σ} and Sp^{O} has closed symm mon structure.

S-mod, Sp^{Σ}, Sp^{O} are Quillen equivalent.
(so that they share iso. homotopy cat).

invert w.e. to the homotopy cat.
smash product $\rightsquigarrow$ derived smash product $\wedge^{\mathbb{L}}$

$E \wedge^{\mathbb{L}} F := CE \wedge CF$ where C is the cofibrant replacement.

S is cofibrant. so it is also the unit of $\wedge^{L}$ in SH.

Good properties: Triangulated.
stable: fiber sequence = cofiber sequence.

- ring spectra. (is like H-space).

monoid in SH. in other words. E is a ring spectrum if
$\exists$ a multiplication $\mu: E\wedge E \to E$. and unit $S\wedge E \to E$

commute up to homotopy.

Warning: we can take monoids in Sp^{Σ}/Sp^{O} then pass to SH. But this would be too strict.

ring spectra $\longleftrightarrow$ multiplicative cohomology theories.

Ex). MU. KU.

- richer ring structure. (see more in foling's talk).

What thing you exchange during weddings and commute?

A commutative ring!

What thing you exchange during weddings and almost commute?

An E_∞-ring!

we might want more multiplicative structures. encode by operad.

In particular, there is a family called E_∞-operad.

Ex). little disk operad.

Robinson defines E_n-stage (commute up to n-dim).

MU, KU, $\mathbb{S}$ are E_∞-ring spectra.

non ex) $(MU)^{\wedge}_p \simeq V\Sigma BP$.

BP is E_4.