# Complex Orientation & Formal Groups

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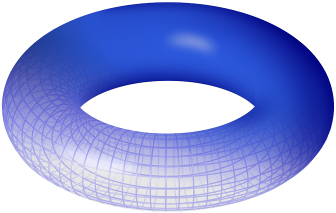
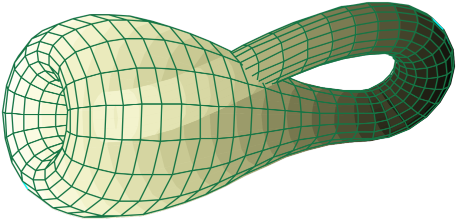
# Orientations on manifolds





- Orientation on manifold: continuous choice of normal vectors
- Let $M$ be a $n$-dim, closed manifold. $x \in M$.
- Normal vectors at $x \iff$ generators of $H_n(M, M - x) \cong \mathbb{Z}$.

**Definition**

An **orientation** on $M$ is an element in $H_n(M)$ which is sent to a generator by the induced map $H_n(M) \to H_n(M, M - x) \cong \mathbb{Z}$ for any $x \in M$.

# Orientations on vector bundles

- Let $V \to M$ is an $n$-dim vector bundle.
- Let $V_x$ be the ventor space over $x \in M$.
  "Local orientation" $\iff H_n(V_x, V_x - x) \cong \mathbb{Z}$.
- Notice that $H_n(V_x, V_x - x) \cong \widetilde{H}_n(Th(V_x \to \{x\}))$
  natural inclusion $Th(V_x) \hookrightarrow Th(V)$

## Definition

An **orientation** on $V \to M$ is an element in $\widetilde{H}^n(Th(V))$ which is sent to generator by the induced map $\widetilde{H}^n(Th(M)) \to \widetilde{H}^n(Th(V_x)) \cong \mathbb{Z}$ for any $x \in M$. Such element is called a **Thom class**.

## Thom isomorphism

Assume that $V \to M$ is oriented with Thom class $c \in \widetilde{H}^n(Th(V))$. The following composition is an isomorphism:

$$H^*(M) \cong H^*(V) \xrightarrow{c \smile (-)} H^{*+n}(V, V - M) \cong \widetilde{H}^{*+n}(Th(V))$$

# $E$-orientation

Let $E$ be a commutative ring spectrum.

## Definition

An $E$-**orientation** on $V \to M$ is an element in $\widetilde{E}^n(Th(V))$ which is sent to generator (as $\pi_0 E$-modules) by the induced map

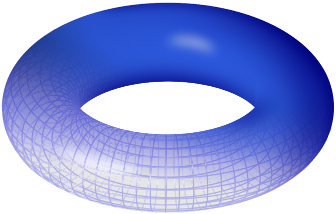
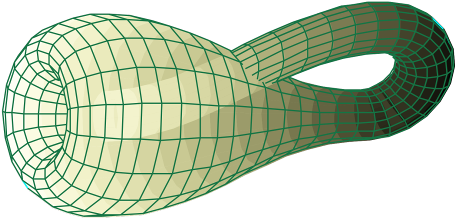
$$\widetilde{E}^n(Th(M)) \to \widetilde{E}^n(Th(V_x)) = \widetilde{E}^n(S^n) \cong \pi_0 E$$

for any $x \in M$.

Classical orientation $=$ $H\mathbb{Z}$-orientation

# Some remarks

- Orientation on a manifold = orientation on its normal vector bundle.

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- Torus: $\mathbb{Z}$-orientable & $\mathbb{Z}/2$-orientable
  Klein bottle: not $\mathbb{Z}$-orientable, but $\mathbb{Z}/2$-orientable
- In fact, all connected real manifolds are $\mathbb{Z}/2$-orientable.
- $\implies H^*(-;\mathbb{Z}/2)$ is “orientable”?

# Complex orientation

Idea: $E$ is complex orientable = "all complex vector bundles are $E$-orientable" + "universal choice of $E$-orientation"

## Definition

A **complex orientation** on $E$ consists of one element $c_V \in \widetilde{E}^{2n}(Th(V))$ for any $n \in \mathbb{Z}^+$ and rank $n$ complex vector bundle $V \to M$, such that

**(1)** For any $x \in M$, $c_V$ is mapped into a generator by

$$\widetilde{E}^{2n}(Th(V)) \to \widetilde{E}^{2n}(Th(V_x)) \cong \widetilde{E}^{2n}(S^{2n}) \cong \pi_0 E$$

**(2)** For any map $f : N \to M$, $f^*(c_V) = c_{f^*V}$.
**(3)** For different complex bundles $V_1, V_2$ over $X$, $c_{V_1 \oplus V_2} = c_{V_1} \cdot c_{V_2}$.

# Special complex bundles

- $\mathbb{C}P^n =$ "complex lines" in $\mathbb{C}^{n+1}$
- **Tautological bundle**: $\mathcal{O}_n := \{(l, z) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} : z \in l\}$.
  The projection $\mathcal{O}_n \to \mathbb{C}P^n$ becomes a complex line bundle.
- $Th(\mathcal{O}_n) \simeq \mathbb{C}P^{n+1}$
- $\mathbb{C}P^0 \hookrightarrow \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \ldots$ induces
  $\mathcal{O}_0 \hookrightarrow \mathcal{O}_1 \hookrightarrow \mathcal{O}_2 \hookrightarrow \ldots$, which is exactly
  $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3 \hookrightarrow \ldots$
- $E$-Thom classes form a sequence in
  $\ldots \to \widetilde{E}^2(Th(\mathcal{O}_n)) \to \widetilde{E}^2(Th(\mathcal{O}_{n-1})) \to \ldots \to \widetilde{E}^2(Th(\mathcal{O}_0)) \to \pi_0 E$
- which agrees with
  $\ldots \to \widetilde{E}^2(\mathbb{C}P^{n+1}) \to \widetilde{E}^2(\mathbb{C}P^n) \to \ldots \to \widetilde{E}^2(\mathbb{C}P^1) \xrightarrow{\simeq} \pi_0 E$
- Let $n \to \infty$, we obtain an element in $\widetilde{E}^2(\mathbb{C}P^\infty)$, which is sent to a generator by $\widetilde{E}^2(\mathbb{C}P^\infty) \to \widetilde{E}^2(\mathbb{C}P^1) \cong \pi_0 E$

# An alternative definition

**Theorem**

There is a natural bijection between
**(1)** Complex orientations of $E$
**(2)** Classes $u \in \widetilde{E}^2(\mathbb{C}P^\infty)$ which restricts to an $\pi_0 E$-module generator of $\widetilde{E}^2(\mathbb{C}P^1) \cong \pi_0 E$.

- Examples:
- Ordinary cohomology $H\mathbb{Z}$:
  $\widetilde{H}^2(\mathbb{C}P^\infty) \cong \widetilde{H}^2(\mathbb{C}P^1) \cong \mathbb{Z}$
- Complex $K$-theory $KU$
  $KU^0(X) = \{\text{isomorphism classes of complex vector bundles over } X\}$.
  Bott periodicity: $KU^{*+2}(X) = KU^*(X)$,
  $KU^0(\mathbb{C}P^1) = \mathbb{Z}[\mathcal{O}_1]/(\mathcal{O}_1 - 1)^2$.
  The generator of $\widetilde{KU}^0(\mathbb{C}P^1)$ is represented by $\mathcal{O}_1 - 1$.
  Choose $\mathcal{O}_\infty - 1 \in \widetilde{KU}^0(\mathbb{C}P^\infty)$ as the complex orientation.
  (choice not unique)

# Obstruction theory

- Recall: complex orientation = pre-image of a generator through $\widetilde{E}^2(\mathbb{C}P^\infty) \to \widetilde{E}^2(\mathbb{C}P^1)$
- $\iff$ extending $\Sigma^\infty \mathbb{C}P^1_+ \to \Sigma^{-2}E$ to a map $\Sigma^\infty \mathbb{C}P^\infty_+ \to \Sigma^{-2}E$
- $\iff$ extending the map from $\mathbb{C}P^1$ to maps from $\mathbb{C}P^2, \mathbb{C}P^3, \ldots$
- $\mathbb{C}P^2 = \mathbb{C}P^1 + 4$ cell
  cofiber sequence $S^3 \to \mathbb{C}P^1_+ \to \mathbb{C}P^2_+$
- Apply maps to $\Sigma^{-2}E$:
  $\ldots \leftarrow \pi_5 E \leftarrow \widetilde{E}^2(\mathbb{C}P^1) \leftarrow \widetilde{E}^2(\mathbb{C}P^2) \leftarrow \ldots$
- **Obstruction** in $\pi_5 E$: extension always exists if $\pi_5 E = 0$
- Similarly, the obstruction of extending a map from $\mathbb{C}P^n$ to a map from $\mathbb{C}P^{n+1}$ is in $\pi_{2n+3}E$.
- **Theorem:** $E$ is complex orientable if its odd degree homotopy groups are trivial.
- Examples: $\pi_* H\mathbb{Z} = \mathbb{Z}$, $\pi_* KU = \mathbb{Z}[u^\pm]$ with $|u| = 2$.

# Formal groups

When further studying the structure of $\widetilde{E}^2(\mathbb{C}P^\infty)$, we obtain some algebraic object called a formal group.

## Definition

A **formal group (law)** over a ring $R$ is a power series $F(x, y) \in R[[X, Y]]$ such that
**(1)** $F(x, y) = F(y, x)$
**(2)** $F(x, 0) = F(0, x) = x$
**(3)** $F(F(x, y), z) = F(x, F(y, z))$

- Let $F_1, F_2$ be formal groups over $R$. A map of formal groups $f : F_1 \to F_2$ is a power series $f(x) \in R[[x]]$ such that $f(0) = 0$ and $f(F_1(x, y)) = F_2(f(x), f(y))$.
- $f$ is an isomorphism if and only if $f'(0)$ is a unit.
- Examples: Additive formal group $G_a(x, y) = x + y$
  Multiplicative formal group $G_m(x, y) = x + y + xy$

# Formal groups & complex orientations

## Atiyah-Hirzebruch spectral sequence

For any space $X$ and cohomology theory $E$, there exists a spectral sequence

$$E_2^{p,q} = H^p(X; E^q(*)) \Rightarrow E^{p+q}(X)$$

- Input: $H^*(X)$, $\pi_* E \xrightarrow{AHSS}$ Output: $E^*(X)$
- It works when $E$ is complex orientable & $X$ is a finite CW complex.
- When $X = \mathbb{C}P^n$, $E^*(\mathbb{C}P^n) = (\pi_* E)[t]/(t^{n+1})$
- Let $n \to \infty$, $E^*(\mathbb{C}P^\infty) = (\pi_* E)[[t]]$
  We can choose $t \in \widetilde{E}^2(\mathbb{C}P^\infty)$ as the given complex orientation.
- Same computation on $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$:
  $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = (\pi_* E)[[x, y]]$
- Here $x, y$ can be chosen as $p_1^*(t)$ and $p_2^*(t)$.
  $p_1, p_2 : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ are the projections.

# Formal groups & complex orientations

- There is a natural product on $\mathbb{C}P^\infty$:
- $\mathbb{C}P^\infty \simeq BU(1)$, which classifies complex line bundles.
- There is a map $m : BU(1) \times BU(1) \to BU(1)$ representing tensor product between line bundles.
- $L_1, L_2$ are line bundles over $X$ classified by $f_1, f_2 : X \to BU(1)$. Then $L_1 \otimes L_2$ is classified by $X \xrightarrow{f_1 \times f_2} BU(1) \times BU(1) \xrightarrow{m} BU(1)$
- Apply the $E$-cohomology: $m^* : (\pi_* E)[[t]] = E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = (\pi_* E)[[x, y]]$
- Define a formal group by $F(x, y) = m^*(t)$.
- Different choices of orientations will give different, but isomorphic formal groups.

# Example on $KU$

- Consider $E = KU$ with the canonical orientation $\mathcal{O}_\infty - 1$
- Notice that $\mathcal{O}_\infty \to \mathbb{C}P^\infty = BU(1)$ is the universal line bundle.
- Thus the map $BU(1) \xrightarrow{\Delta} BU(1) \times BU(1) \xrightarrow{m} BU(1)$ classifies $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$
- Consider the map $m^* : (\pi_* E)[[t]] = E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = (\pi_* E)[[x, y]]$
- $m^*(\mathcal{O}_\infty) = \mathcal{O}_\infty \otimes \mathcal{O}_\infty$
- $m^*(t + 1) = (x + 1)(y + 1) \Longrightarrow m^*(t) = x + y + xy$
- Multiplicative formal group $G_m$ on $KU$

# Conclusion

- Local orientation $\Longleftrightarrow$ local (co)homology
- Global orientation $\Longleftrightarrow$ $H_n(M)$ or $H^n(Th(V))$
- $E$-orientation $=$ universal choice of $E$-orientations on all complex bundles
- $\Longrightarrow$ determined by all tautological line bundles
- $\Longrightarrow$ represented by an element in $\widetilde{E}^2(\mathbb{C}P^\infty)$
- Formal group encodes the (co)product on $E^*(\mathbb{C}P^\infty)$
- Coming next: universal example $MU$, more about formal groups