

# $E_*(MU)$ & $MU$ as a universal COCT

iWocit 2022

- Calculations on  $E_*(BU)$  &  $E^*(BU)$   $E \in \text{COCT}$   
 $E_*(BU) = E_*(\beta_1, \beta_2, \dots)$   
 $E^*(BU) = E^*([c_1, c_2, c_3, \dots])$
- COCT revisit &  $E_*(MU)$   
 $E_*(MU) = E_*(b_1, b_2, \dots)$
- $MU$  as a universal example

## Main references

- Kochman's book  
Bordism, stable homotopy & Adams SS
- Adams's blue book  
Stable homotopy & generalised homology
- Lurie's lecture notes
- Switzer's book  
Algebraic Topology, Homotopy & Homology

§I  $E_*(BU)$  &  $E^*(BU)$

$E \in \text{COCT}$

Recall  $\odot H_*(BU)$  &  $H^*(BU)$

$$H_*( ) = H_*( ; \mathbb{Z} )$$

$$H^*( ) = H^*( ; \mathbb{Z} )$$

Thm (Kochman's book section 2.3)

$C_i$ :  $i$ -th chem class.

$$H^*(BU(m)) \cong \mathbb{Z}[C_1, \dots, C_n] \quad |C_i| = 2i$$

$$m < n \quad BU(m) \longrightarrow BU(n)$$

$$H^*(BU(m)) \longrightarrow H^*(BU(n))$$

$$C_i \longmapsto \begin{cases} C_i & i \leq m \\ 0 & i > m \end{cases}$$

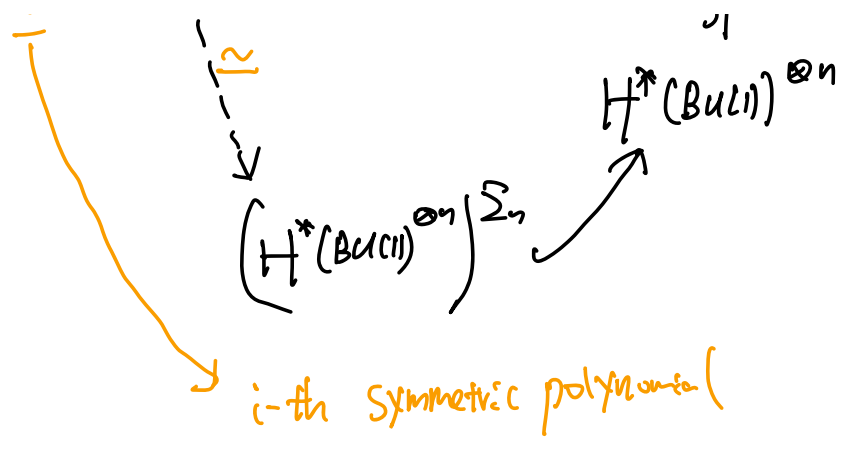
$$BU(m) \times BU(n) \xrightarrow{\oplus} BU(m+n)$$

$$H^*(BU(m+n)) \longrightarrow H^*(BU(m)) \otimes H^*(BU(n))$$

$$\begin{array}{ccc} C_{m+n} & \xrightarrow{\quad} & C_m \otimes C_n \\ \downarrow \gamma_i & \xrightarrow{\quad} & \downarrow \gamma_n \\ BU(i) \times \dots \times BU(i) & \xrightarrow{\oplus n} & BU(n) \end{array}$$

$$H^*(BU(m)) \xrightarrow{\quad} H^*(BU(1)) \times \dots \times H^*(BU(1))$$

$C_i \quad 1 \quad \quad \quad 1 \quad \quad \quad 1$



- $H_*(BU(n))$  is a free  $\mathbb{Z}$ -module on monomials

$$\frac{\beta_{i_1} \beta_{i_2} \dots \beta_{i_n}}{H_*(\mathbb{C}P^\infty) \simeq \mathbb{Z} \{ 1, \beta_1, \beta_2, \dots \}}$$

$\beta_0$   
 $\parallel$

- UCT:

$$H^*(BU(n)) \xrightarrow{\sim} \text{Hom}(H_*(BU(n)); \mathbb{Z})$$

$$c_i \longmapsto (\beta_i)^* : H_*(BU(n)) \longrightarrow \mathbb{Z}$$

$$\begin{cases} \beta_i \longmapsto 1 \\ \text{other} \longmapsto 0 \end{cases}$$

$$H^*(BU(\infty)) \stackrel{?}{\simeq} \varinjlim H^*(BU(n))$$

Cor :

$$\bullet H^*(BU) \cong \mathbb{Z}[C_1, C_2, \dots] \quad |C_i| = 2i$$

$$\underline{\text{Lim}^1 = 0}$$

$$H^*(BU(n)) \xrightarrow{\otimes E} H^*(BU(n+1)) \otimes G$$

$$E^*(BU(n)) \xrightarrow{\otimes E} E^*(BU(n+1))$$

$$\bullet H_*(BU) \cong \mathbb{Z}[\beta_1, \beta_2, \dots] \quad |\beta_i| = 2i$$

• Both are Hopf algebras.

$$\psi(C_n) = \sum_{k=0}^n C_k \otimes C_{n-k}$$

$$\psi(\beta_n) = \sum_{k=0}^n \beta_k \otimes \beta_{n-k}$$

## Recall ② pairing in AHSS

Thm: Let  $E$  be a ring spectrum,  $X$  be a finite CW-complex

We have homological & cohomological AHSS

$$E_{n,t}^2 = H_n(X; E^t) \Rightarrow E_{n,t}^{ht}(X)$$

$$E_2^{n,t} = H^n(X; E^t) \Rightarrow E^{n+1}(X)$$

Then there is a pairing

$$\langle -, - \rangle : E_r^{n,s} \otimes E_{n,t}^r \longrightarrow E_{s+t}$$

St

- pairing on  $E_2$ -page is the pairing on singular (co)-homology with coefficients

$$H^n(X; G) \otimes H_n(X; G') \longrightarrow G \otimes G'$$

by evaluation

- on Ev-page:  $\langle d^r x, y \rangle \simeq \langle x, d^{r^*} y \rangle$

- The pairing on  $E^*(X) \otimes E_*(X)$  induces the pairing on  $E_\infty \otimes E^\infty$

$$\langle \cdot, \cdot \rangle : E^n(X) \otimes E_m(X) \rightarrow E_{m-n}$$

$f, g$

$$\langle f, g \rangle : S^m \rightarrow E \wedge X \rightarrow E \wedge \Sigma^n E \rightarrow \Sigma^n E$$

- Actually for convergence issue we need the spectrum to be bounded below, i.e.,  $\pi_*(E)$  is bounded below. The proof of the following results actually works on bounded below spectra. However, the results still hold on general case.

See Lurie's Lecture 4 notes.

Ex:  $E_*(\mathbb{C}P^\infty)$  is a COCT

Apply AHSS to compute  $E_*(\mathbb{C}P^n)$

$$E_2 = H_*(\mathbb{C}P^n; E_*) \xrightarrow[\text{collapse on } E_2]{\Rightarrow} E_*(\mathbb{C}P^n)$$

$$H^*(\mathbb{C}P^n; E^*) \xrightarrow[\text{collapse on } E_2]{\Rightarrow} E^*(\mathbb{C}P^n)$$

The pairing shows the AHSS collapses on

$E_2$ -page

$$E_*(\mathbb{C}P^n) = E_* \{1, \beta_1, \dots, \beta_n\}$$

$$\Rightarrow E_*(\mathbb{C}P^\infty) = E_* \{1, \beta_1, \beta_2, \dots\} \quad |\beta_i| = 2i$$

$E$ : COCT

Thm:  $E$  is a COCT

- $E_*(BUCM)$  is a free  $E_*$ -module on monomials

$$\beta_{i_1} \cdots \beta_{i_n}$$

corner-polyd-chem classes

- $E^*(BUCM) = E_*[\underbrace{c_1, \dots, c_n}]$

$$E^*(BUCM) \xrightarrow{\sim} \text{Hom}_{E_*}(E_*(BUCM), E_*)$$

$$c_i \longmapsto (\beta_i)^*$$

- Similar properties about  $H^*(BUCM)$  listed above also hold on  $E^*(BUCM)$

e.g.  $E^*(BUCM) \twoheadrightarrow E^*(BUCM+1)$

$$c_i \longmapsto \begin{cases} c_i & i \leq n-1 \\ 0 & i = n \end{cases}$$



Sketch proof:

- Apply homological ATSS

$$H_* (BU(n); E_*) \implies E_* (BU(n))$$

Claim: this SS collapses on  $E_2$ -page

Consider

$$\begin{array}{ccc}
 BU(1) \times \dots \times BU(1) & \xrightarrow{\oplus n} & BU(n) \\
 \beta_{i_1} \otimes \beta_{i_2} \otimes \dots \otimes \beta_{i_n} & \xrightarrow{\quad} & \beta_{i_1} \dots \beta_{i_n} \\
 H_* (BU(1))^{*\otimes n} \otimes E_* & \xrightarrow{\quad} & H_* (BU(n)) \otimes E_* \\
 \downarrow \text{SI} & & \downarrow \text{SI} \\
 H_* (BU(1) \times \dots \times BU(1); E_*) & \longrightarrow & H_* (BU(n); E_*) \\
 \downarrow & & \downarrow \\
 E_* (BU(1) \times \dots \times BU(1)) & & E_* (BU(n))
 \end{array}$$

$$\leadsto E_* (BU(n)) \simeq H_* (BU(n)) \otimes E_*$$

- $sk_q(BU(n))$ : subcomplex of  $BU(n)$  with  $\dim \leq q$

Apply cohomological AHSS

$$H^*(sk_q BU(n); E^*) \xRightarrow[\text{collapses on } E_2]{} E^*(sk_q BU(n))$$

claim: This SS also collapse on  $E_2$ -page

Consider the homological AHSS

$$H_*(sk_q BU(n); E_*) \xRightarrow[\text{collapses on } E_2]{} E_*(sk_q BU(n))$$

- The pairing is non-singular

Therefore  $E^*(\text{Sk}_q BU(n)) \simeq H^*(\text{Sk}_q BU(n)) \otimes E_*$

pass to  $BU(n)$

$$E^*(BU(n)) \simeq E_*[\underbrace{c_1, c_2, \dots, c_n}]$$

- those generators depends on the choice of the complex orientation of  $E$

Cor:  $E$  a COCT

$$E_* BU \simeq E_*[\beta_1, \beta_2, \dots] \quad |\beta_i| = 2i$$

$$E^* BU \simeq E_*[c_1, c_2, \dots] \quad |c_i| = 2i$$

## §II COCT revisited & $E_*(MU)$

Thm: TFAE

1)  $E$  is a complex oriented ring spectrum

$$\begin{array}{ccc} E^2(\mathbb{C}P^\infty) & \longrightarrow & E^2(\mathbb{C}P^1) \\ u^1 \longmapsto & & \mathbb{1} \end{array}$$

2) each complex bundle is  $E$ -oriented

$$\begin{array}{ccc} V \rightarrow \tilde{X} & u \in E^n(\text{Th}(V)) & \\ \downarrow & \downarrow & \\ \mathbb{1} & E^n(S^2) & \end{array}$$

Sketch proof:

2)  $\implies$  1) ✓

Assume 2) is true. Then the tautological line bundle  $\gamma_1 \rightarrow BU(1)$  is  $E$ -oriented

$$\begin{array}{ccc} E^2(\text{Th}(\gamma_1)) & \longrightarrow & E^2(S^2) \\ u_{\gamma_1} \longmapsto & & \mathbb{1} \end{array}$$

$$1) \implies 2)$$

it suffices to show that each fundamental bundle  $\gamma_n \rightarrow BU(n)$  is  $E$ -oriented

$$\begin{array}{ccc} \gamma_1 \oplus \dots \oplus \gamma_1 & \longrightarrow & \gamma_n \\ \downarrow \uparrow & & \downarrow \\ BU(1) \times \dots \times BU(1) & \xrightarrow{\oplus n} & BU(n) \end{array}$$

Apply Thom space functor:

$$\begin{array}{ccc} Th(\gamma_1) \wedge \dots \wedge Th(\gamma_1) & \longrightarrow & Th(\gamma_n) \\ MU(1) \wedge \dots \wedge MU(1) & \longrightarrow & MU(n) \end{array}$$

- $E^*(MU(n)) \simeq E_*[C_n]$

this is because  $MU(n) \simeq BU(n)/BU(n-1)$

Disk bundle  $EU(n) \times_{//} D^{2n} \simeq EU(n)/U(n) = BU(n)$

Sphere bundle  $EU(n) \times_{\alpha(n)} S^{2n-1} \simeq EU(n)/U(n) = BU(n-1)$

$$\begin{array}{ccc}
 \bullet \quad E^*(MU(n)) & \longrightarrow & E^*(MU(n)) \otimes_{E'} \dots \otimes_{E'} E^*(MU(n)) \\
 & & \downarrow \\
 C_n & \longmapsto & \chi^E \otimes \dots \otimes \chi^E
 \end{array}$$

$C_n$  is a Thom class in  $E^{2n}(MU(n))$

i.e.  $\gamma_n$  is  $E$ -oriented

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Thom isomorphism:

Thm:

if a  $n$ -dim bundle  $V \rightarrow X$  is  $E$ -oriented, we

have the following isomorphism

$$\underline{E_{*+n}(Th(V))} \xrightarrow{\sim} E_*(X)$$

Cor:

$$E_{*+2n}(MU(n)) \xrightarrow{\sim} E_*(BU(n))$$

$$E_*(\Sigma^{2n} MU(n)) \xrightarrow{\sim} E_*(BU(n))$$

$$MU = \text{hocolim } \Sigma^{\infty-2n} MU(n) \quad \left\{ \begin{array}{l} BU = \text{colim } BU(n) \end{array} \right.$$

$$E_*(MU) \xrightarrow{\sim} E_*(BU)$$

Then: There is a ring isomorphism

$$\Phi: E_*(MU) \xrightarrow{\sim} E_*(BU)$$

$$\text{i.e. } E_*(MU) \simeq E_*[b_1, b_2, \dots]$$

$$\text{where } \Phi(b_i) = \beta_i$$

- See Switzer's book chapter 16 for details
- $MU$  is complex oriented

$$MU_* MU \simeq MU_* [b_1, b_2, \dots]$$

fact:  $(MU_*, MU_* MU)$  is a Hopf algebra!

# § III MU as a universal COCT

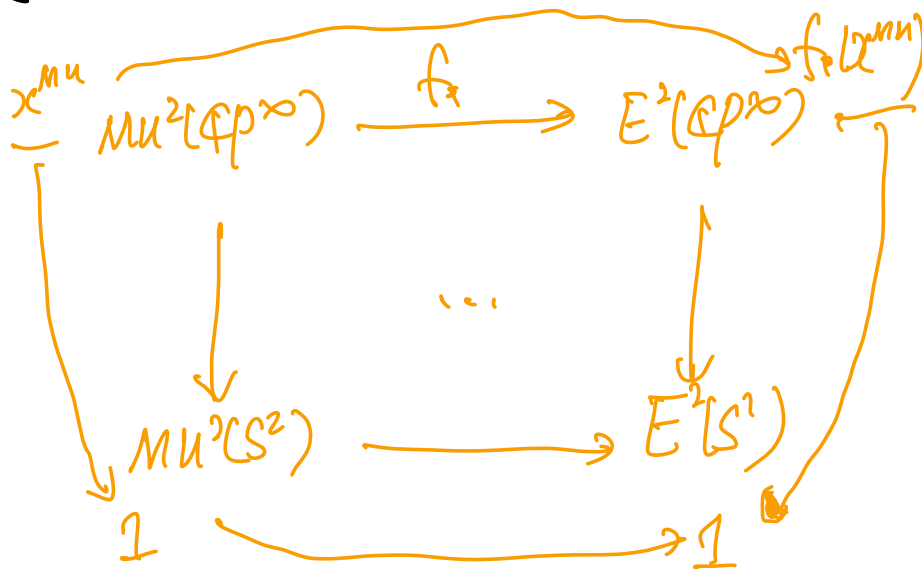
Thm: There is a one-by-one correspondence

$$\left\{ \begin{array}{l} \text{ring maps} \\ \text{MU} \rightarrow E \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{complex orientations} \\ \text{on } E \end{array} \right\}$$

Sketch proof:

$$\mathcal{F} = \left\{ \begin{array}{l} \text{ring maps} \\ \text{MU} \rightarrow E \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{complex orientations} \\ \text{on } E \end{array} \right\}$$

$$f: \text{MU} \rightarrow E \longmapsto f_*(\alpha^{\text{MU}})$$





$$\Phi: \left. \begin{array}{l} \text{complex orientations} \\ \text{on } E \end{array} \right\} \longrightarrow \left. \begin{array}{l} \text{ring maps} \\ MU \rightarrow E \end{array} \right\}$$

$$\chi^E \longmapsto \Phi(\chi^E)$$

How to construct a ring map

$$\Phi(\chi^E): MU \longrightarrow E \quad ?$$

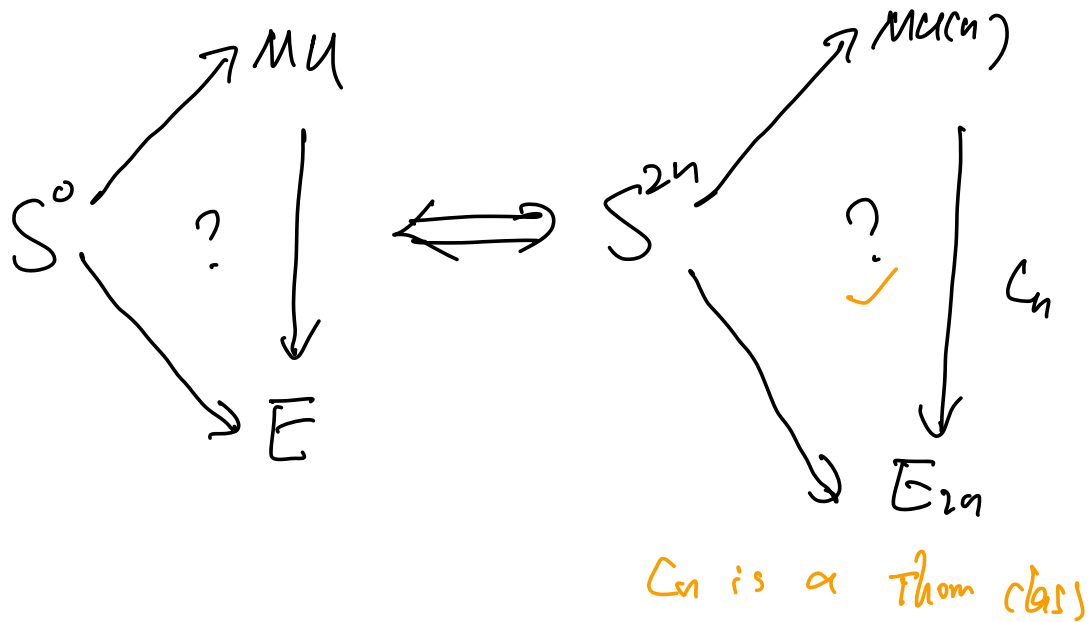
point-set level:  $E^*(MU(n)) \simeq E_{\mathbb{F}}[[C_n]]$

$$C_n: MU(n) \longrightarrow E_{2n}$$

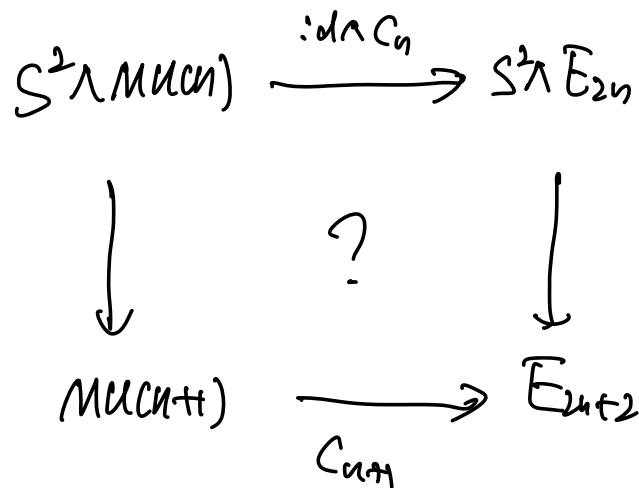
claim: the collection  $\{C_n\}$  gives a ring map

$$\Phi(\chi^E): MU \longrightarrow E$$

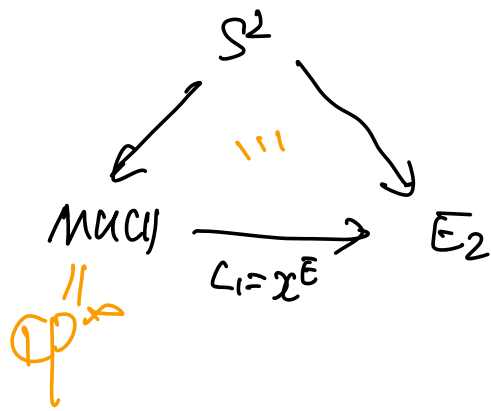
① Unity



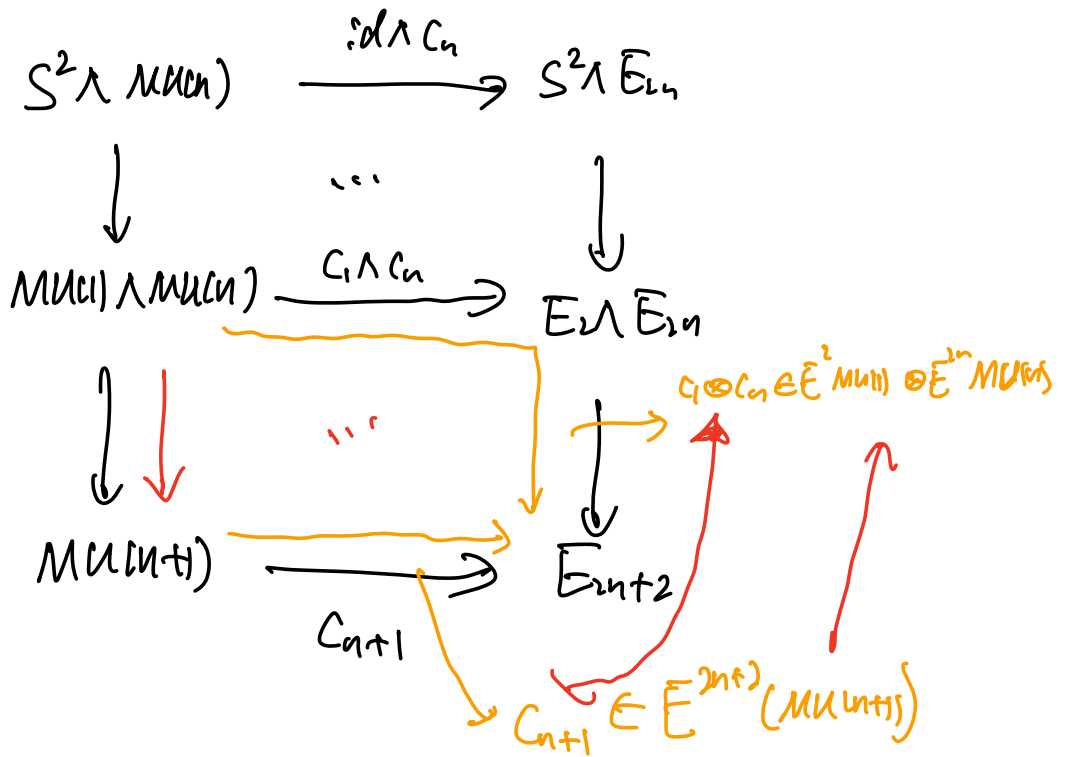
② Structure map



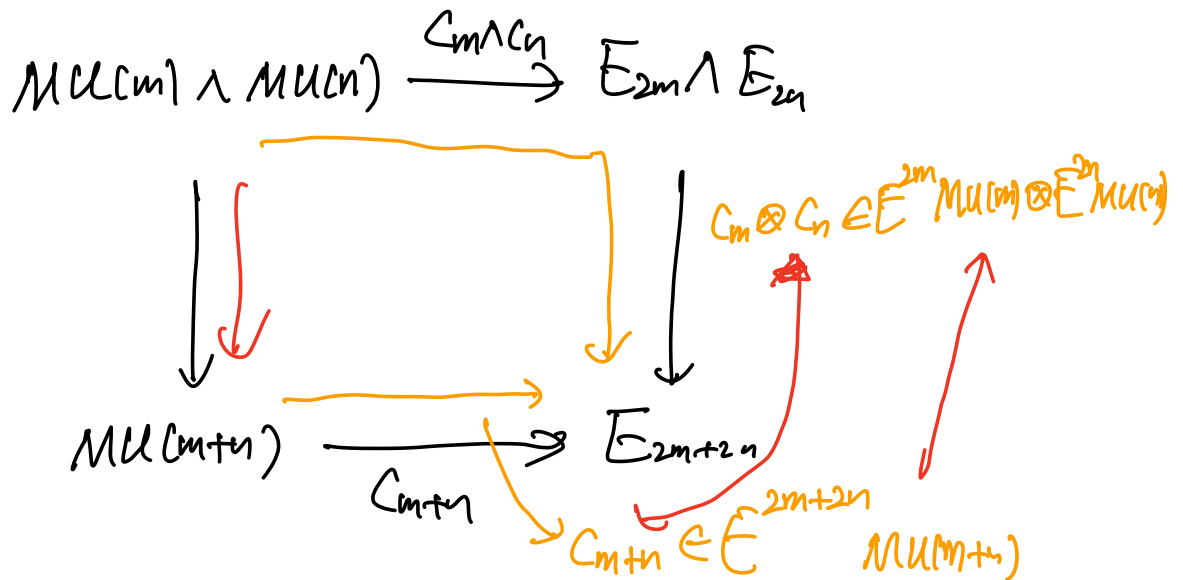
when  $n=0$



In general



③ Ring structure



- The two maps are inverse to each other

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- The FGL  $f_{MU}$  determined by  $(MU, \chi^{MU})$  is a universal one in the sense that for any given

$f \in FGL/R_*$ ,  $\exists !$  ring map  $\varphi: MU_* \rightarrow R_*$

st  $\varphi_*(f_{MU}) = f$

See more discussions about (MUs, MUs, MU)  
in next two lectures tomorrow!