$E_*(MU)$ & MU as a universal COCT

iWoat 2022

- Calculations on $E_*(BU)$ & $E^*(BU)$ $E \in$ COCT

  $E_*(BU) = E_*[\beta_1, \beta_2, \cdots]$

  $E^*(BU) = E_*[[c_1, c_2, c_3, \cdots]]$

- COCT revisit & $E_*(MU)$

  $E_*(MU) = E_*[b_1, b_2, \cdots]$

- MU as a universal example

Main reference

- Kochman's book

  Bordism, stable homotopy & Adam SS

- Adam's blue book

  stable homotopy & generalised homology

- Lurie's lecture notes

- Switzer's book

  Algebraic Topology, Homotopy & Homology

§I $E_*(BU)$ & $E^*(BU)$ $E \in \mathrm{CoCT}$

Recall ① $H_*(BU)$ & $H^*(BU)$

$H_*(\ ) = H_*(\ ;\mathbb{Z})$

$H^*(\ ) = H^*(\ ;\mathbb{Z})$

Thm (Kochman's book section 2.3)

c_i : i-th chern class.

- $$H^*(BU(n)) = \mathbb{Z}[c_1, \cdots, c_n] \qquad |c_i| = 2i$$

- $m < n$ $\quad BU(m) \longrightarrow BU(n)$

$$H^*(BU(n)) \twoheadrightarrow H^*(BU(m))$$

$$c_i \longmapsto \begin{cases} c_i & i \le m \\ 0 & i > m \end{cases}$$

- $$\begin{array}{ccc} \gamma_m \oplus \gamma_n & & \\ \downarrow \quad \downarrow & & \\ BU(m) \times BU(n) & \xrightarrow{\oplus} & BU(m+n) \end{array}$$

$$H^*(BU(m+n)) \longrightarrow H^*(BU(m)) \otimes H^*(BU(n))$$

$$c_{m+n} \longmapsto c_m \otimes c_n$$

Σ_n

$$\begin{array}{ccc} \gamma_1 \oplus \cdots \oplus \gamma_1 & & \gamma_n \\ \downarrow \qquad \downarrow & & \downarrow \\ BU(1) \times \cdots \times BU(1) & \xrightarrow{\oplus n} & BU(n) \end{array}$$

$$H^*(BU(n)) \longrightarrow H^*(BU(1) \times \cdots \times BU(1))$$

c_i $\qquad \qquad \qquad \qquad \cong$

$$\left(H^*(BU(1))^{\otimes n}\right)^{\Sigma_n} \longrightarrow H^*(BU(1))^{\otimes n}$$

i-th symmetric polynomial

- $H_*(BU(n))$ is a free $\mathbb{Z}$-module on monomials $\underline{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_n}}$

$$H_*(\mathbb{CP}^\infty) \simeq \mathbb{Z}\{\underset{\substack{\| \\ \beta_0}}{1}, \beta_1, \beta_2, \cdots\}$$

- UCT:

$$H^*(BU(n)) \xrightarrow{\sim} \mathrm{Hom}(H_*(BU(n)); \mathbb{Z})$$

$$c_i \longmapsto (\beta_1^i)^*: H_*(BU(n)) \longrightarrow \mathbb{Z}$$

$$\begin{cases} \beta_1^i \longmapsto 1 \\ \text{other} \longmapsto 0 \end{cases}$$

$$H^*(BU \overset{\text{Def}}{=} \operatorname{colim}_n BU(n)) \overset{?}{=} \lim H^*(BU(n))$$

Cor :

- $H^*(BU) \simeq \mathbb{Z}[c_1, c_2, \cdots]$ $\quad |c_i| = 2i$

$\lim^1 = 0$ $\qquad H^*(BU(n)) \otimes E \twoheadrightarrow H^*(BU(n-1)) \otimes G$

$\cong$

$E^*(BU(n)) \twoheadrightarrow E^*(BU(n-1))$

- $H_*(BU) \simeq \mathbb{Z}[\beta_1, \beta_2, \cdots]$ $\quad |\beta_i| = 2i$

- Both are Hopf algebras.

$$\psi(c_n) = \sum_{k=0}^{n} c_k \otimes c_{n-k}$$

$$\psi(\beta_n) = \sum_{k=0}^{n} \beta_k \otimes \beta_{n-k}$$

Recall ⊗ pairing in AHSS

Thm: Let E be a ring spectrum, X be a finite CW-cpx

we have homological & cohomological AHSS

$$E^2_{n,t} = H_n(X; E_t) \Rightarrow E_{n+t}(X)$$

$$E_2^{n,t} = H^n(X; E^t) \Rightarrow E^{n+t}(X)$$

Then there is a pairing

$$\langle -,- \rangle : E_r^{n,-s} \otimes E^r_{n,t} \longrightarrow E_{s+t}$$

s.t.

- pairing on E_2-page is the pairing on singular (co)-homology with coefficients

$$H^n(X; G) \otimes H_n(X; G') \longrightarrow G \otimes G'$$

by evaluation

- on E_r-page: $\langle d_r x, y\rangle = \langle x, d^r y\rangle$

- The pairing on $E^*(X)\otimes E_*(X)$ induces the pairing on $E_\infty \otimes E^\infty$

$\langle -,- \rangle \quad E^n(X)\otimes E_m(X) \longrightarrow E_{m-n}$

$f, \quad g$

$$\langle f, g\rangle : \begin{tikzcd} S^m \arrow[r] \arrow[rrr, bend right] & E\wedge X \arrow[r] & E\wedge \Sigma^n E \arrow[r] & \Sigma^n E \end{tikzcd}$$

- Actually for convergence issue we need the spectrum to be bounded below, i.e, $\pi_*(E)$ is bounded below. The proof of the following results actually works on bounded below spectra. However, the results still hold on general case. See Lurie's Lecture 4 notes.

Ex: $E_*(\mathbb{CP}^\infty)$ E a COCT

Apply AHSS to compute $E_*(\mathbb{CP}^n)$

$$E_2 = H_*(\mathbb{CP}^n; E_*) \xRightarrow[\text{collapse on } E_2]{} E_*(\mathbb{CP}^n)$$

$$\Uparrow$$

$$H^*(\mathbb{CP}^n; E^*) \xRightarrow[\text{collapses on } E_2]{} E^*(\mathbb{CP}^n)$$

The pairing shows the AHSS collapses on E_2-page

$$E_*(\mathbb{CP}^n) = E_*\{1, \beta_1, \dots, \beta_n\}$$

$$\Longrightarrow E_*(\mathbb{CP}^\infty) = E_*\{1, \beta_1, \beta_2, \dots\} \quad |\beta_i| = 2i$$

E: COCT

Thm: E is a COCT

- $E_*(BU(n))$ is a free E_*-module on monomials

$\beta_{i_1} \cdots \beta_{i_n}$

conner-floyd-chern classes

- $E^*(BU(n)) = E_*[[\underbrace{c_1, \cdots, c_n}]]$

$$E^*(BU(n)) \xrightarrow{\sim} \mathrm{Hom}_{E_*}(E_*(BU(n)); E_*)$$

$$c_i \longmapsto (\beta_1^i)^*$$

- Similar properties about $H^*(BU(n))$ listed above also hold on $E^*(BU(n))$

e.g $E^*(BU(n)) \longrightarrow E^*(BU(n-1))$

$$c_i \longmapsto \begin{cases} c_i & i \leq n-1 \\ 0 & i = n \end{cases}$$

Sketch proof:

- Apply homological AHSS

$$H_*(BU(n); E_*) \Longrightarrow E_*(BU(n))$$

Claim: this ss collapses on E_2-page

Consider $\quad BU(1)\times\cdots\times BU(1) \xrightarrow{\oplus n} BU(n)$

$$\begin{tikzcd}
\beta_{i_1}\otimes\beta_{i_2}\otimes\cdots\otimes\beta_{i_n} \arrow[r, squiggly] & \beta_{i_1}\cdots\beta_{i_n} \\
H_*(BU(1))^{\otimes n}\otimes E_* \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & H_*(BU(n))\otimes E_* \arrow[d, phantom, "\cong" rotate=-90] \\
H_*(BU(1)\times\cdots\times BU(1); E_*) \arrow[r] \arrow[d, Rightarrow] & H_*(BU(n); E_*) \arrow[d, Rightarrow] \\
E_*(BU(1)\times\cdots\times BU(1)) & E_*(BU(n))
\end{tikzcd}$$

$\rightsquigarrow E_*(BU(n)) \simeq H_*(BU(n))\otimes E_*$

- $sk_q(BU(n))$: subcomplex of $BU(n)$ with $\dim \leq q$

Apply cohomological AHSS

$$H^*(sk_q BU(n) ; E^*) \underset{\substack{\text{collapses} \\ \text{on } E_2}}{\Longrightarrow} E^*(sk_q BU(n))$$

Claim: This SS also collapse on E_2-page

Consider the homological AHSS

$$H_*(sk_q BU(n) ; E_*) \underset{\substack{\text{collapses} \\ \text{on } E_2}}{\Longrightarrow} E_*(sk_q BU(n))$$

- The pairing is non-singular

Therefore $E^*(sk_q BU(n)) \simeq H^*(sk_q BU(n)) \otimes E_*$

pass to $BU(n)$

$$E^*(BU(n)) \xrightarrow{\sim} E_*[[c_1, c_2, \dots, c_n]]$$

- those generators depends on the choice of the complex orientation of E

Cor: E a COCT

$$E_* BU \simeq E_*[\beta_1, \beta_2, \dots] \qquad |\beta_i| = 2i$$

$$E^* BU \simeq E_*[[c_1, c_2, \dots]] \qquad |c_i| = 2i$$

§II COCT revisit & $E_*(MU)$

Thm: TFAE

1) E is a complexed oriented ring spectrum

$$\begin{aligned} E^2(\mathbb{CP}^\infty) &\longrightarrow E^2(\mathbb{CP}^1) \\ u &\longmapsto 1 \end{aligned}$$

2) each complex bundle is $\underset{\sim\sim\sim\sim\sim\sim\sim}{E\text{-oriented}}$

$V \to X$ $\quad u \in E^n(Th(V))$

$$\begin{aligned} u \in E^n&(Th(V)) \\ \downarrow \quad & \quad \downarrow \\ 1 \quad E^n&(S^n) \end{aligned}$$

Sketch proof:

2) $\Longrightarrow$ 1) ✓

Assume 2) is true. Then the tautological line bundle $\gamma_1 \longrightarrow BU(1)$ is E-oriented

$$\begin{aligned} E^2(Th(\gamma_1)) &\longrightarrow E^2(S^2) \\ u_{\gamma_1} &\longmapsto 1 \end{aligned}$$

1) $\Longrightarrow$ 2)

it suffices to show that each tautological bundle $\gamma_n \longrightarrow BU(n)$ is E-oriented

$$\begin{tikzcd} \gamma_1 \oplus \cdots \oplus \gamma_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \gamma_n \arrow[d] \\ BU(1) \times \cdots \times BU(1) \arrow[r, "\oplus n"] & BU(n) \end{tikzcd}$$

Apply Thom space functor:

$$\begin{array}{ccc} Th(\gamma_1) \wedge \cdots \wedge Th(\gamma_1) & \longrightarrow & Th(\gamma_n) \\ MU(1) \wedge \cdots \wedge MU(1) & \longrightarrow & MU(n) \end{array}$$

- $E^*(MU(n)) \simeq E_*[[c_n]]$

this is because $MU(n) \simeq BU(n)/BU(n-1)$

Disk bundle $\quad EU(n) \times_{U(n)} D^{2n} \simeq EU(n)/U(n) = BU(n)$

Sphere bundle $EU(n) \times_{U(n)} S^{2n-1} \simeq EU(n)/U(n-1) = BU(n-1)$

- $E^*(MU(n)) \longrightarrow E^*(MU(1)) \otimes_{E^*} \cdots \otimes_{E} E(MU(1))$

$$c_n \longmapsto x^E \otimes \cdots \otimes x^E \mapsto 1$$

- c_n is a thom class in $E^{2n}(MU(n))$

i.e γ_n is E-oriented #

Thom isomorphism:

Thm:
If a n-dim bundle $V \to X$ is E-oriented, we have the following isomorphism

$$\underline{E_{*+n}}(Th(V)) \xrightarrow{\sim} E_*(X)$$

Cor:

$$E_{*+2n}(MU(n)) \xrightarrow{\simeq} E_*(BU(n))$$

$$E_*(\Sigma^{-2n} MU(n)) \xrightarrow{\simeq} E_*(BU(n))$$

$MU = \operatorname{hocolim} \Sigma^{\infty-2n} MU(n)$ $\qquad$ $BU = \operatorname{colim} BU(n)$

$$E_*(MU) \xrightarrow{\simeq} E_*(BU)$$

Thm: There is a ring isomorphism

$$\Phi: E_*(MU) \xrightarrow{\simeq} E_*(BU)$$

i.e $\quad E_*(MU) \simeq E_*[b_1, b_2, \cdots]$

where $\quad \Phi(b_i) = \beta_i$

- See Switzer's book chapter 16 for details
- MU is complex oriented

$$MU_* MU \simeq MU_*[b_1, b_2, \cdots]$$

Fact: $(MU_*, MU_* MU)$ is a Hopf algebroid!

§III MU as a universal COCT

Thm: There is a one-by-one correspondence

$$\left\{\begin{array}{c}\text{ring maps}\\ MU \to E\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{complex orientations}\\ \text{on } E\end{array}\right\}$$

Sketch proof:

$$\varphi : \left\{\begin{array}{c}\text{ring maps}\\ MU \to E\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{complex orientations}\\ \text{on } E\end{array}\right\}$$

$$f : MU \to E \longmapsto f_*(x^{MU})$$

$$\begin{tikzcd}
x^{MU} \arrow[rrr, bend left, mapsto] \arrow[d, phantom, "\in" description] & & & f_*(x^{MU}) \arrow[d, phantom, "\in" description] \\
MU^2(\mathbb{C}P^\infty) \arrow[rrr, "f_*"] \arrow[d] & & & E^2(\mathbb{C}P^\infty) \arrow[d] \\
& \cdots & & \\
MU^2(S^2) \arrow[rrr] & & & E^2(S^2) \\
1 \arrow[u, phantom, "\in" description] \arrow[rrr, bend right, mapsto] & & & 1 \arrow[u, phantom, "\in" description]
\end{tikzcd}$$

$$\Phi : \left\{ \begin{array}{l} \text{complex orientations} \\ \text{on } E \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{ring maps} \\ MU \to E \end{array} \right\}$$

$$x^E \longmapsto \Phi(x^E)$$

How to construct a ring map

$$\Phi(x^E) : MU \longrightarrow E \quad ?$$

point-set level: $E^*(MU(n)) \simeq E_*[[c_n]]$

$$c_n : MU(n) \longrightarrow E_{2n}$$

claim: the collection $\{c_n\}$ gives a ring map

$$\Phi(x^E) : MU \longrightarrow E$$

① Unity

$$\begin{tikzcd} & MU \arrow[d] \\ S^0 \arrow[ur] \arrow[r, phantom, "?"] \arrow[dr] & E \\ & E \end{tikzcd} \iff \begin{tikzcd} & MU(n) \arrow[dd, "c_n"] \\ S^{2n} \arrow[ur] \arrow[r, phantom, "?"] \arrow[dr] & \\ & E_{2n} \end{tikzcd}$$

c_n is a Thom class

② structure map

$$\begin{tikzcd} S^2 \wedge MU(n) \arrow[r, "id \wedge c_n"] \arrow[d] \arrow[dr, phantom, "?"] & S^2 \wedge E_{2n} \arrow[d] \\ MU(n+1) \arrow[r, "c_{n+1}"'] & E_{2n+2} \end{tikzcd}$$

when $n=0$

$$
\begin{tikzcd}
 & S^2 \arrow[dl] \arrow[dr] & \\
MU(1) \arrow[rr, "c_1 = x^E"'] & & E_2
\end{tikzcd}
$$

$MU(1) = \mathbb{CP}^\infty$

In general

$$
\begin{tikzcd}
S^2 \wedge MU(n) \arrow[r, "id \wedge c_n"] \arrow[d] & S^2 \wedge E_{2n} \arrow[d] \\
MU(1) \wedge MU(n) \arrow[r, "c_1 \wedge c_n"] \arrow[d] & E_2 \wedge E_{2n} \arrow[d] \\
MU(n+1) \arrow[r, "c_{n+1}"'] & E_{2n+2}
\end{tikzcd}
$$

$c_1 \otimes c_n \in \tilde{E}^2 MU(1) \otimes \tilde{E}^{2n} MU(n)$

$c_{n+1} \in \tilde{E}^{2n+2}(MU(n+1))$

③ ring structure

$$
\begin{tikzcd}
MU(m) \wedge MU(n) \arrow[r, "C_m \wedge C_n"] \arrow[d] & E_{2m} \wedge E_{2n} \arrow[d] \\
MU(m+n) \arrow[r, "C_{m+n}"'] & E_{2m+2n}
\end{tikzcd}
$$

$C_m \otimes C_n \in E^{2m} MU(m) \otimes E^{2n} MU(n)$

$C_{m+n} \in E^{2m+2n} MU(m+n)$

- The two maps are inverse to each other

#

- The FGL f_{MU} determined by (MU, x^{MU}) is a universal one in the sense that for any given f FGL$/R_*$, $\exists\,!$ ring map $\varphi : MU_* \longrightarrow R_*$ s.t. $\varphi_*(f_{MU}) = f$

See more discussions about (MLE, MCMC)

in next two Lectures tomorrow!