Quillen's Theorem on $\pi_* MU$

Review · Yutao's lecture

{complex oriented cohomology theories} $\rightsquigarrow$ {formal group laws}

$$E^*(\mathbb{CP}^\infty) \cong \pi_* E[[t]] \qquad\qquad F(x,y) = f^*(t)$$
$$\downarrow f^* \qquad\qquad \longmapsto \qquad\qquad \in \pi_* E[[x,y]]$$
$$E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_* E[[x,y]]$$

Zhipeng's lecture.

MU is universal among complex oriented cohomology theories.

i.e. if E is a complex oriented theory,
then $f: MU \to E$. map of ring spectra,
st $E^*(\mathbb{CP}^\infty) \cong \pi_* E[[t_E]]$
where $t_E = f^*(t_{MU})$.

Q: Is the formal group law associated to MU "universal"?

A: Yes. Quillen's theorem on $\pi_* MU$.

NOT the historical order! "Astonishing Discovery"

The universal formal group law F over the Lazard ring L.

Theorem: $\theta : L \to \pi_* MU$ is an isomorphism.

"pf"

$$\begin{array}{ccc} L & \xrightarrow{\theta} & \pi_* MU \\ \text{①}\downarrow & \text{②} & \downarrow\text{③} \\ \mathbb{Z}[b_1, b_2, \cdots] & = & H_* MU \end{array}$$

① Universal formal group law over the Lazard ring L.

Recall that a fgl. F/R : $F(x,y) \in R[[x,y]]$

st. $F(x,0) = x$

$F(x,y) = F(y,x)$

$F(x, F(y,z)) = F(F(x,y), z)$.

$$\left\{ x+y+\sum_{i,j\geq 1} a_{ij} x^i y^j \;\middle|\; a_{ij} \cdots \right\}$$

$F(x,y) = x+y+\sum_{i,j\geq 1} a_{ij} x^i y^j$ over $R[a_{ij}]/\sim = L$

i.e. if G is a fgl. over R'

then $\exists\, \theta : L \to R'$

st $G = f_*(G)$

i.e. $G(x,y) = x+y+\sum \theta(a_{ij}) x^i y^j$

$|x| = |y| = -2$ $a_{ij} = 2(i+j-1)$

Thm (Lazard) $L \cong \mathbb{Z}[x_1, x_2, \cdots]$ $|x_i| = 2i$ $i > 0$

Lurie 2, 3, Green book $\left\{ \begin{array}{l} A2.1.10, \\ A2.1.12. \end{array} \right\}$

Complex oriented cohomology theory & Stable homot groups

Prop. $\phi: \mathcal{L} \to \mathbb{Z}[b_1, b_2, \cdots]$ after $\otimes \mathbb{Q}$ is an iso.

Observation: ex of fgls. $F(x,y) = x+y$.

given $g(x) = x + b_1 x^2 + b_2 x^3 + \cdots$

$g(x)$ is invertible in $\mathbb{Z}[b_1, \cdots][[x]]$

$g\,F(g^{-1}(x), g^{-1}(y))$ is a fgl $/ \mathbb{Z}[b_1, \cdots]$.

$g(g^{-1}(x) + g^{-1}(y))$ / ———

Then $\phi: L \to \mathbb{Z}[b_1, b_2, \cdots]$

Fact. in characteristic zero, every fgl. is obtained from the additive formal grp law $f(x,y) = x+y$ by a change of variables $g(t) = x + \sum b_i x^{i+1}$.

Next lecture

As a consequence,

$\phi \otimes \mathbb{Q}$ is isomorphism

$I \subset L$ with positive degree

Define. $I = (b_1, \cdots)$

Then. $(I/I^2)_{2n} \hookrightarrow (J/J^2)_{2n} \cong \mathbb{Z}$ (L3) indecomposable part.

image $\mathbb{Z}$ else.

$p\mathbb{Z}$ $i = p^k - 1$

i.e.

$$\begin{array}{l} L \cong \mathbb{Z}[x_1, \dots] \\ \phi \downarrow \\ \mathbb{Z}[b_1, b_2, \dots] \end{array}$$

$$x_i \to \begin{cases} p b_i & i = p^k - 1 \\ b_i & \text{else.} \end{cases}$$

(2). $\mathbb{Z}[b_1, \dots]$ $\begin{cases} f\,g\,l \text{ as above. } g(t) = t + b_1 t^2 + \dots \\ H_* MU \text{ from } \mathbb{Z}\text{hipeng}. \end{cases}$

$$H_*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}\{\beta_0, \beta_1, \dots\}$$

$$\beta_i \longleftrightarrow t^i$$

$$H_*(BU(n); \mathbb{Z}) = \mathbb{Z}\{\beta_{i_1} \cdots \beta_{i_n}\}$$

$0 \le i_1 \le \dots \le i_n$

Explain $\mathbb{Z}[b_1, \dots] \cong H_*(MU, \mathbb{Z})$.

More general. E cplx. $E^*(MU(1)) \simeq E \cdot (\pi_2 E)[\![t]\!]$

then π_* $E^* \mathbb{CP}^\infty$

$b_i \subseteq E_* MU(1) \hookrightarrow t^{i+1}.$

From the fgl. $E_* MU = \pi_* E \wedge MU$

two complex orientations t_E. t_{MU}.

$\pi_* E[b_1, \dots][[t_E]] \cong (E \wedge MU)^* (\mathbb{CP}^\infty)$

$\cong \pi_* E[b_1, \dots b_n \dots][[t_{MU}]]$

$\rightsquigarrow$. $t_{MU} = \sum a_i t_E^{i+1}$

$a_i \in \pi_* E[b_1, \dots]$

Claim $a_i = b_i$ f?

Pf. t_{MU} : $MU(1) \to MU \wedge E$

E-module map $\downarrow$:

$\underline{MU(1) \wedge E} \to MU \wedge E$ $\quad MU(1) \wedge E \simeq \bigvee_{i \geq 0} \Sigma^{2i} E$

$(MU(1) \hookrightarrow MU)) \wedge E$.

t_E : $MU(1) \wedge E \to E \to MU \wedge E$.

? t^i : $MU(1) \wedge E \xrightarrow{proj} \Sigma^{2i} S \wedge E$.

b_i : $S^{2i} \wedge E \to MU \wedge E$.

(b) $L = \mathbb{Z}[x_1, \dots] \xrightarrow{f} \pi_* MU$ Milnor. $= \mathbb{Z}[a_1, a_2, \dots]$

$\phi \downarrow \qquad \downarrow h$

$\mathbb{Z}[b_1, \dots] = H_* MU$.

Adam Spectral Sequence.

$\mathrm{Ext}^s_{A_*}(\mathbb{F}_2, H_t MU) \Rightarrow \pi_{t-s} MU$.

[Next lecture] A_* dual Steenrod Algebra.

Ext or comodule

Zhipeng: $H_*MU = \mathbb{Z}[b_1, \dots]$

Need A_*-comodule structure.

$\parallel$

$H\mathbb{F}_p \wedge H\mathbb{F}_p = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_0, \tau_1, \dots]$

$|\xi_i| = 2p^i - 2 \quad |\tau_i| = 2p^i - 1$

odd p

For $p=2$. $\mathbb{F}_p[\zeta_1, \dots]$

$|\zeta_i| = 2^i - 1$

"$\mathbb{F}_p[\zeta_1^2, \dots] \otimes E[\zeta_i, \dots]$"

P_*

Hopf Alg: Alg + coAlg

Green Book Appendix 1.

$$\Delta \xi_n = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i \qquad \xi_0 = 1$$

$$\Delta \tau_n = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i$$

H_*MU is a $P_* = \mathbb{F}_p[\xi_1, \dots]$-comodule.

Moreover, $\cong P_* \otimes \mathbb{F}_p[u_1, \dots]$

$|u_i| = 2i$ for $i \neq p^k - 1$

relation 足够多

$$C = \bar{\mathbb{F}}_p[u_1, \cdots]$$
$$= \mathbb{F}_p \square_{P_*} H_*MU$$

$$\mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*MU) = \mathrm{Ext}_{A_*}(\mathbb{F}_p, P_* \otimes C)$$
$$= \mathrm{Ext}_{A_*}(\mathbb{F}_p, P_* \otimes E \underset{E}{\square} C)$$
$$= \mathrm{Ext}_{A_*}(\mathbb{F}_p, A_* \underset{C}{\square} C)$$
$$= \mathrm{Ext}_E(\mathbb{F}_p, C)$$

E acts trivially on C evenly
odd dim.

$$\Rightarrow \quad \mathrm{Ext}_E(\mathbb{F}_p, C) = \mathrm{Ext}_E(\mathbb{F}_p, \mathbb{F}_p) \otimes C$$
$$\cong \mathbb{F}_p[\epsilon_j] \quad j \geq 0 \quad |\epsilon_j| = (p^j - 1, 1)$$

Thm MU.

$i_{k>0}$.

lecture 2.

L9

Review. $A_* = H\mathbb{F}_2 \wedge H\mathbb{F}_2$.

(strict) Automorphism of the additive fgl.

$\{ f(x) = x + a_1 x^2 + a_2 x^4 + \cdots \}$

$\circlearrowright$

$HMU_* = \mathbb{Z}[b_1, \cdots]$

$g(y) = y + b_1 y^2 + b_2 y^3 + \cdots$

! warning the standard action is from

$$f(x) = x + a_1 x^2 + a_2 x^4 + \cdots$$

$$\xi_1 \qquad \xi_2$$

$H^*(\mathbb{C}P^\infty) = \mathbb{F}_2[y] \hookrightarrow H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[x]$

$f_1(y) = (f(x))^2 = (x^2 + a_1^2 x^4 + \cdots)$.

prop?

$Gr(F) \to G_1 \xrightarrow{F} G'$

$E \cdot E \quad A \quad P$

$P \otimes C_* \cong H_* MU$

Change of ring:

Module. $H^*(G, M) = \mathrm{Ext}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$

$M = \mathbb{Z}[G] \otimes_{\mathbb{Z}[N]} N$

$K \to G \to G/K$

$\Sigma = \mathrm{Spec}(H^* MU)$ $\quad H^* C$

$\Sigma/G \qquad (\Sigma/\ker F)/G' \qquad \Sigma \times B\ker(F)/G$

$\Sigma_0 \times B\ker F$

$\mathbb{F}_2 L \qquad \alpha \qquad \pi \; \mathbb{R}P^\infty$

FGL.

ex. $F_a(x, y) = x + y$

$F(x, y) = x + y + xy$

$(1 + F) = (1+x)(1+y)$

$\log(1+F) = \log(1+x) + \log(1+y)$

$\log(1+x) = \sum_{i>0} (-1)^{i+1} \frac{x^i}{i} \qquad \mathbb{Q}[[x]]$