Quillen's Thm on $\pi_* MU$

Review: Yutao's lecture

$$\{\text{complex oriented cohomology theories}\} \rightsquigarrow \{\text{formal group laws}\}$$

$$E^*(\mathbb{CP}^\infty) \cong \pi_* E[[t]] \qquad\qquad F(x,y) = f^*(t)$$
$$\downarrow f^* \qquad\qquad \mapsto \qquad\qquad \in \pi_* E[[x,y]]$$
$$E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_* E[[x,y]]$$

Zhipeng's lecture.

MU is universal among complex oriented cohomology theories

i.e., if E is a cplx oriented cohomology theory
then $\exists\ f: MU \to E$ map of ring spectra
st $E^*(\mathbb{CP}^\infty) = \pi_* E[[t_E]]$
$t_E = f^*(t_{MU})$

Q: Is the fgl associated to MU "universal"?

Yes! Quillen's thm on $\pi_* MU$.

Not the historical order. astonishing!!!

The universal fgl over L is studied by Lazard!

. .

⇝ Lazard ring.

Thm. $\theta : L \longrightarrow \pi_* MU$ is an isomorphism

outline of pf.

$$\begin{array}{ccc} L & \longrightarrow & \pi_* MU \\ ① \downarrow & & \downarrow ③ \\ \mathbb{Z}[b_1, b_2, \cdots] & \cong & H_* MU \\ & ② & \end{array}$$

①. universal fgl over L

Recall that a fgl F over a commutative ring R

is $F(x,y) \in R[[x,y]]$

st

$$\left.\begin{array}{l} F(x,0) = x \\ F(x,y) = F(y,x) \\ F(x, F(y,z)) = F(F(x,y),z) \end{array}\right\} (*)$$

$$F(x,y) = x+y + \sum_{i,j \geq 1} a_{ij} x^i y^j$$

requirement of $\{a_{ij}\}$

universal one : F_{univ} over L.

i.e., given any fgl G over R.

then $\exists$ ring morphism, $f : L \to R$.

st. $G = f_*(F_{univ})$.

$$L = \mathbb{Z}[a_{ij}] / \sim_{(*)} \qquad |x| = |y| = -2$$

$$F_{univ}(x,y) = x+y + \sum a_{ij} x^i y^j \qquad |a_{ij}| = 2(i+j-1)$$

$i,j \geq 1$

Thm. (Lazard) $L \cong \mathbb{Z}[x_1, x_2, \cdots]$

$|x_i| = 2i \qquad i > 0.$

Reference: Lurie's notes on Chromatic homotopy theory.
Lecture 2 & 3.
(L2 & L3)

A2.1.10

Doug Ravenel "Green book" A2.1.12
Complex cobordism theory
and stable homotopy groups
of sphere.

Prop $\phi : L \to \mathbb{Z}[b_1, b_2, \cdots]$
$\phi \otimes \mathbb{Q}$ is an isomorphism.

observation: ex of fgls $\quad f(x,y) = x+y$

Given $g(x) = x + b_1 x^2 + b_2 x^3 + \cdots$
($g(x)$ is invertible in $\mathbb{Z}[b_1, b_2, \cdots][[x]]$).

Then. $gF(g^{-1}(x), g^{-1}(y))$ is also
a fgl.

In particular, $g(g^{-1}(x) + g^{-1}(y))$

over $\mathbb{Z}[b_1, b_2, \dots]$

$\rightsquigarrow$ $\phi : L \to \mathbb{Z}[b_1, b_2, \dots]$

Fact: in char $= 0$, every fgl is obtained from the additive one $F(x, y) = x + y$ by a change of coordinates $g(t) = x + \sum l_i x^{i+1}$

$\rightsquigarrow$ indicates that $\phi \otimes \mathbb{Q}$ is an iso

Harder Fact: Define $I \underset{\text{ideal}}{\subset} L$ consists of elements of positive degrees,

$J \subset \mathbb{Z}[b_1, \dots]$
$= (b_1, b_2, \dots)$

indecomposable part.

$$\begin{array}{c} (I/I^2)_{2n} \\ \downarrow \phi \\ (J/J^2)_{2n} \cong \mathbb{Z} \cdot \{b_n\} \end{array}$$

ϕ is an injection

the image is $p \cdot \mathbb{Z}$ if $n = p^k - 1$
$\mathbb{Z}$ else.

②. $\mathbb{Z}[b_1, b_2, \cdots]$ $\uparrow$ fgl $g(t) = t + b_1 t^2 + \cdots$

$H_* MU$

($H_*(\mathbb{CP}^\infty) = \mathbb{Z}\{\beta_0, \beta_1, \cdots\}$

$\beta_i \leftrightarrow t^i$
dual

$H^*(\mathbb{CP}^\infty) = \mathbb{Z}[t]$)

$H_* MU = \mathbb{Z}[b_1, b_2, \cdots]$

More general case.

$E_* MU = \pi_* E[b_1, b_2, \cdots]$

$\underbrace{\pi_*(E \wedge MU)}$ ← has two orientations

t_E : $MU \to E \xrightarrow{\simeq} E \wedge S^0 \to E \wedge MU$.

t_{MU} : $MU \xrightarrow{id} MU = S^0 \wedge MU \to E \wedge MU$.

$\underline{\underline{\pi_*(E \wedge MU)[[t_E]]}} \cong (E \wedge MU)^*(\mathbb{CP}^\infty) \cong \pi_*(E \wedge MU)[[t_{MU}]]$

$\pi_* E_*[b_1, b_2, \cdots]$

$\Rightarrow$ $t_{MU} = t + \sum a_i t_E^{i+1}$ $a_i \in \pi_* E[b_1, \cdots]$

Claim: $a_i = b_i$ [?].

Pf: $t_{MU} \leftrightarrow \phi_{MU}: MU(1) \to MU \wedge E$.

$\downarrow$

E-module $MU(1) \wedge E \to MU \wedge E$.

$$MU(1) \wedge E$$

$$(MU(1) \hookrightarrow MU) \wedge E$$

Advantage: $MU(1) \wedge E \simeq \bigvee_{i \geq 0} \Sigma^{2i} E$

$t_E \leftrightarrow \phi_E : MU(1) \to E \to MU \wedge E$

$\int MU(1) \wedge E \to MU \wedge E$ E-module

b_i: $S^{2i} \wedge E \xrightarrow{\text{E-module}} MU \wedge E$

$$MU(1) \wedge E \xrightarrow{\text{proj}} \Sigma^{2i} E \xrightarrow{b_i + t_E} MU \wedge E.$$

$$\Rightarrow \phi_M = \Sigma\, t^i \cdot b_i \cdot \phi_E.$$

$$t_{mu} = t + \Sigma\, b_j t^{j+1}$$

③ $H_* MU \rightsquigarrow \pi_* MU.$

Adams Spectral sequence.

$$\mathrm{Ext}^s_{A_*}(\mathbb{F}_p, H_* MU) \Rightarrow \pi_{t-s} MU^\wedge_p$$

Alg $\rightsquigarrow$ Top

A_* dual of Steenrod Alg.

$$= H\mathbb{F}_{p*} H\mathbb{F}_p = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes E[\tau_0, \tau_1, \ldots]$$

$|\xi_i| = 2p^i - 2 \quad |\tau_i| = 2p^i - 1$

Greenlook $H_* MU = \mathbb{Z}[b_1, b_2, \ldots] \qquad P_* \otimes E_*.$

A1.1.4. Definition. *Let M and N be right and left Γ-comodules, respectively. Their* cotensor product over Γ *is the K-module defined by the exact sequence*

$$0 \to M \,\square_\Gamma\, N \to M \otimes_A N \xrightarrow{\psi\otimes N - M\otimes\psi} M \otimes_A \Gamma \otimes_A N,$$

where ψ denotes the comodule structure maps for both M and N.

Structure. H_*MU is a P_*-comodule.

$\cong \mathbb{F}_p[u_i]_{i \neq p^i-1} \otimes P_*$

A1.1.18. Corollary. *Let (K, Σ) be a commutative graded connected Hopf algebra over a field K. Let M be a K-algebra and a right Σ-comodule and let $C = M \,\square_\Sigma\, K$. If there is a surjection $f\colon M \to \Sigma$ which is a homomorphism of algebras and Σ-comodules, then M is isomorphic to $C \otimes \Sigma$ simultaneously as a left C-module and as a right Σ-comodule.* □

Change of ring

$\mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*MU)$

$\cong \mathrm{Ext}_{E_*}(\mathbb{F}_p, C) \cong \mathrm{Ext}_{E_*}(\mathbb{F}_p, \mathbb{F}_p) \otimes C$

A1.3.13. Corollary. *Let K be a field and $f\colon (K,\Gamma) \to (K,\Sigma)$ ~~be a~~ surjective map of Hopf algebras. If N is a left Σ-comodule then*

$$\mathrm{Ext}_\Gamma(K, \Gamma \,\square_\Sigma\, N) = \mathrm{Ext}_\Sigma(K, N).$$ □

E_2-page. ASS for MU

filtration 0 filtration !

$\mathbb{F}_p[u_i, \epsilon_j]$ ←

$i \neq p^k - 1$

$\Rightarrow$. $\pi_* MU = \mathbb{Z}[x_1, x_2, \ldots]$ $\quad |x_i| = 2i$

$\downarrow$

$H_* MU = \mathbb{Z}[b_1, \ldots]$

$x_i \to b_i \qquad i \neq p^k - 1$

$x_j \to p b_i \qquad j = p^k - 1$

θ:

$$\begin{array}{ccc}
L & \longrightarrow & \pi_* MU \\
\shortparallel & & \shortparallel \\
\mathbb{Z}[x_1, x_2, \ldots] & \longrightarrow & \mathbb{Z}[y_1, y_2, \ldots] \\
\downarrow \otimes \mathbb{Q} \ \cong & & \downarrow \otimes \mathbb{Q} \ \cong
\end{array}$$

$\mathbb{Z}[b_1, \ldots, b_n] = H_* MU$.

$(I/I^2)_{2n} \qquad (J/J^2)_{2n}$.

$x_i = y_i \qquad i \neq p^k - 1$

$\downarrow \qquad \downarrow$

$b_i = b_i$

$b_i = b_i$

$$\begin{array}{ccc} x_i & \longrightarrow & y_i \\ \downarrow & & \downarrow \\ p b_i & \longrightarrow & p b_i \end{array} \qquad i = p^k - 1$$

Steenrod alg.

$H^* \mathbb{CP}^\infty$ is an A - module

$\uparrow$

Hopf Algebra.

$E^* \mathbb{CP}^\infty$ $\qquad E^*E$ - module.

$\uparrow$

Hopf Algebra.