Morava K & E Theories

inaat 2022

§0 Some higher algebra

Analog:

ordinary algebra

Sp <- - - - - - - - -> Ab

(com) ring sp
module spectra
algebra sp
⋮

(com) ring
modules
algebra
⋮

We need a rigid model for Sp, not just Ho(Sp)

- lose many info

- EKMM S-modules
- Sym/orth spectra
- ∞-cat of Sp.

$\rightsquigarrow$ Ho(Sp)

- a sym monoidal cat of spectra: Sp, $\wedge$

Def: • A ring spectrum is a monoid in Sp

• A commutative ring spectrum is a commutative monoid in Sp.

• we have similar notions in $\mathrm{Ho}(Sp)$

• In point-set level, we have different operads $(E_n)_{n\geq 1}$ to encode higher homotopy coherently

$$\text{ring spectra} \longleftrightarrow \overset{E_1}{\overset{\shortparallel}{A_\infty}}\text{-spectra}$$

$$\text{Com ring spectra} \longleftrightarrow E_\infty\text{-spectra}$$

Def: Let R be a ring spectrum, a (left) R-module is a spectrum M with $\varphi: R\wedge M \to M$ s.t.

$$\begin{tikzcd} S\wedge M \arrow[r, "\text{unit}"] \arrow[dr, "\simeq"'] & R\wedge M \arrow[d] \\ & M \end{tikzcd} \qquad \begin{tikzcd} R\wedge R\wedge M \arrow[r, "\mu_R\wedge \mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\varphi"'] & R\wedge M \arrow[d, "\varphi"] \\ R\wedge M \arrow[r, "\varphi"] & M \end{tikzcd}$$

• $\mathcal{M}_R$: cat of R-modules, complete & complete

let R be a com ring spectrum, $\mathcal{D}_R$ be the derived category of $\mathcal{M}_R$

Def: An R-ring spectrum A is an R-module A with unit $\eta: R \longrightarrow A$ & product $\phi: A \wedge_R A \longrightarrow A$ s.t. the following diagram commutes in $\mathcal{D}_R$

$$\begin{tikzcd}
R \wedge_R A \arrow[r] \arrow[dr, "\cdots"'] & A \wedge_R A \arrow[d, "\phi"] & A \wedge_R R \arrow[l] \arrow[dl, "\cdots"] \\
& A &
\end{tikzcd}$$

A is ass or com if the appropriate diagrams commute

• $\wedge$ should understand as derived $\wedge$.

prop: • If A & B are R-ring spectra, then so is $A \wedge_R B$.

See modern foundation of stable homotopy theory EKMM

• under some mild conditions, cofibers A/x localizations $R[y^{-1}]$ both admit R-ring structures

• $M(k) = \mathrm{Cofib}(\Sigma^{2(p^k-1)} BP \xrightarrow{\cdot v_k} BP)$

• $BP[v_n^{-1}]$

Both are BP-ring spectra.

§I Landweber exact functor thm

$$\mathrm{CoCT} \longrightarrow \mathrm{FGL}$$
$$(E, x^E) \longmapsto F \in E_*[[x,y]]$$

A natural Question: $f: MU_* \to R_*$

Given a formal group law F/R_*, can we find a spectrum s.t. its fgl is F.

A possible candidate: $MU_*(\ ) \otimes_{MU_*} R_*$

If it is a homology thery

$$\begin{array}{ccc} MU_* & \longrightarrow & MU_* \otimes_{MU_*} R_* \\ & \cong & \\ f: MU_* & \longrightarrow & R_* \end{array}$$

- However, it is not a homology thery in general

- Homotopy axiom

- Suspension axiom

- Additivity axiom

- exactness

Given a cofiber sequence $X \to Y \to Z$ in CW-complexes

$$MU_*(X) \otimes_{MU_*} R_* \longrightarrow MU_*(Y) \otimes_{MU_*} R_* \longrightarrow MU_*(Z) \otimes_{MU_*} R_*$$

might not be exact in general

$$MU_*\text{-mod} \longrightarrow Ab \qquad - \otimes_{MU_*} R_*$$
$$M \longmapsto M \otimes_{MU_*} R_*$$

When is it exact ??

- If R_* is a flat MU_*-module

- $MU_* = \pi_*(MU) = \mathbb{Z}[x_1, x_2, \ldots,]$

flat modules $/ MU_*$ are too rare.

- $MU_*(X)$ enjoys a richer structure

Recall: (MU_*, MU_*MU) is a Hopf algebroid.

Def: A (MU_*, MU_*MU)-comodule M is a MU_*-module together with a co-action map

$$M \longrightarrow MU_*MU \otimes_{MU_*} M$$

which is compatible with the comultiplication on MU_*MU.

- $MU_*(X)$ is a (MU_*, MU_*MU)-comodule

the co-action map is induced

$$\begin{array}{c} MU \wedge X \xrightarrow{id \wedge \eta \wedge id} MU \wedge MU \wedge X \\ \cong \\ MU \wedge S^0 \wedge X \end{array}$$

Consider the functor

$$\begin{aligned} (MU_*, MU_*MU)\text{-comod} &\longrightarrow Ab \\ M &\longmapsto M \otimes_{MU_*} R_* \end{aligned}$$

When is this functor exact ??

Recall:

Def: Given a FGL F/R_*, let v_i be the coefficient of x^{p^i} in $[p]_F(x)$.

$= x +_F x +_F \cdots +_F x$

Thm (Landweber)

Let R_* be an MU_*-algebra. Then the functor

$$\begin{aligned} (MU_*, MU_*MU)\text{-comod} &\longrightarrow Ab \\ M &\longmapsto M \otimes_{MU_*} R_* \end{aligned}$$

is exact iff for any p & n, the map

$$R_*/(p, v_1, \dots, v_{n-1}) \xrightarrow{\cdot v_n} R_*/(p, v_1, \dots, v_{n-1})$$

is injective. $(p, v_1, \dots, v_n)$ is a regular sequence in R_*.

In particular, $MU_*(\ ) \otimes_{MU_*} R_*$ is a homology theory.

- If a coct E satisfies this condition

Then we say E is a Landweber theory

p-typical FGL version. $\longleftrightarrow BP_*$

replace 1) FGL by p-typical FGL

2) MU_* by BP_*

(BP_*, BP_*BP) also a Hopf algebroid

$$(BP_*, BP_*BP)\text{-Comod} \longrightarrow Ab$$

$$M \longmapsto M \otimes_{BP_*} R_*$$

is exact iff $\forall n$

$$R_*/(p, v_1, \dots, v_{n-1}) \xrightarrow{\cdot v_n} R_*/(p, v_1, \dots, v_{n-1})$$

is injec.

Ex:

1) $H\mathbb{Z}$ with additive FGL: $F(x,y) = x+y$

$$[p]_F(x) = px \rightsquigarrow \begin{cases} v_0 = p \\ v_i = 0 & i > 0 \end{cases}$$

- $\mathbb{Z}/(p) \xrightarrow{\cdot v_1} \mathbb{Z}/(p)$

is not injective!!

- $H\mathbb{Z}$ is not landweber
- $H\mathbb{Q}$ is landweber: Height 0 theory

2) Complex K-Theory KU with multiplicative FGL

$F = x+y+\beta xy \qquad KU_* \simeq \mathbb{Z}[\beta^{\pm 1}]$

$$[p]_F(x) = \beta^{-1}\left((\beta x+1)^p - 1\right)$$
$$= px + \cdots + \beta^{p-1}x^p \rightsquigarrow \begin{cases} v_0 = p \\ v_1 = \beta^{p-1} \\ v_i = 0 & i > 1 \end{cases}$$

$v_i = 0$

- $$\begin{tikzcd} KU_*/(p) \arrow[r, "\cdot v_1"] \arrow[d, phantom, "\cong" rotate=90] & KU_*/(p) \\ \mathbb{F}_p[\beta^{\pm 1}] \arrow[r, "\cdot \beta^{p-1}"] & \mathbb{F}_p[\beta^{\pm 1}] \end{tikzcd}$$

- $$\begin{tikzcd} KU_*/(p, v_1) \arrow[r, "\cdot v_2"] \arrow[d, phantom, "\shortparallel"] & KU_*/(p, v_1) \arrow[d, phantom, "\shortparallel"] \\ 0 & 0 \end{tikzcd}$$

$\Longrightarrow$ (KU_*, F_*) satisfies the Landweber exactness

$$\Longrightarrow \quad KU_*(X) \simeq MU_*(X) \otimes_{MU_*} KU_*$$

Conner-Floyd thm 1966

3) Johnson-Wilson theory $E(n)$

Let $E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n^{\pm 1}]$

Consider the natural map

gives a p-typical FGL / $E(n)_*$.

$$BP_* \longrightarrow E(n)_*$$

$$v_i \longmapsto \begin{cases} v_i & i \leq n \\ 0 & i > n \end{cases}$$

- $E(n)_*/(p, v_1, \dots, v_i) \xrightarrow{\cdot v_{i+1}} E(n)_*/(p, v_1, \dots, v_i)$

$i < n$ $\quad \mathbb{F}_p[v_{i+1}, \dots, v_n^{\pm 1}] \xrightarrow{\cdot v_{i+1}} \mathbb{F}_p[v_{i+1}, \dots, v_n^{\pm 1}]$

$i \geq n$ $\quad 0 \longrightarrow 0$

So Landweber gives a spectrum $E(n)$ which is height $\leq n$

§2 Morava K-Theory.

Does it exist a Height exactly n complex oriented spectrum?

A naive idea:

$$K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$$

with $\quad BP_* \longrightarrow K(n)_*$

$$\begin{cases} v_n \longmapsto v_n \\ \text{others} \longmapsto 0 \end{cases}$$

However, we cannot construct such spectrum from Landweber exactness. If $n \geq 2$, $v_1 = 0$

$$K(n)_*/(p) \xrightarrow{\cdot v_1} K(n)_*/(p)$$

$$\cong \qquad\qquad\qquad \cong$$

$$0 \qquad\qquad\qquad 0$$

is not inject.

$K(n)_*$ $(F_n)_*$

Def: $K(n) := BP[v_n^{-1}] \wedge_{BP} \bigwedge_{k \neq n} M(k)$ in $\mathcal{D}_{BP}$

- $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$

- $K(n)$ is a homotopy ass ring spectrum

- If $p>2$ $K(n)$ a homotopy commutative ring spectrum

!! not a E_∞-ring spectrum

- $BP_* \longrightarrow K(n)_*$ classifies a height exactly n p-typical FGL. F $[p]_F(x) = v_n x^{p^n}$

Honda's FGL

- $K(1)$ is a summand of mod p complex K-theory KU/p.

prop: • Künneth thm

$$K(n)_*(X\times Y)\simeq K(n)_*(X)\otimes_{K(n)_*}K(n)_*(Y)$$

• let X be a p-local finite CW-complex

Then $K(n)_*(X)=0 \Longrightarrow K(n-1)_*(X)=0$

Thick subcat thm.

§3: Morava E-Theory

Set $(E_n)_* = \mathbb{W}(\mathbb{F}_{p^n})[[u_1,\dots,u_{n-1}]][u^{\pm 1}]$

• $\mathbb{W}(\mathbb{F}_{p^n})$: witt vector of $\mathbb{F}_{p^n}$

complete local ring with maximal ideal (p)

$\mathbb{W}(\mathbb{F}_{p^n})/(p)\simeq\mathbb{F}_{p^n}$

• $|u_i|=0$ $\quad |u|=-2$

Consider $BP_*\longrightarrow(E_n)_*$

$\cdots{}^{p^n}$

$$v_i \longmapsto \begin{cases} u_i u & 0<i<n \\ u^{1-p^n} & i=n \\ 0 & i>n \end{cases}$$

it defines a p-typical FGL $\mathbb{F}/(E_n)_*$ with

$$[p]_{\mathbb{F}}(x) = px +_{\mathbb{F}} \sum^{\mathbb{F}} u_i u^{1-p^i} x^{p^i}$$

- Since u is invertible, the sequence $(p, u_1 u^{1-p}, u_2 u^{1-p^2}, \dots, u^{1-p^n}, 0, \dots)$ satisfies the condition in Landweber

$\Longrightarrow$ a complex oriented spectrum E_n called Morava E-Theory.

Why $(E_n)_*$ looks like this ??

Motivation comes from Deformation theory.

$K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$

$|v_n| = 2(p^n - 1)$

$(K_n)_* = \mathbb{F}_{p^n}[u^{\pm 1}]$

$|u| = -2$

2-periodic Morava K-Theory:

define $\underbrace{(K_n)_* = \mathbb{F}_{p^n}[u^{\pm 1}]}_{\text{ring}}$ $\quad |u| = -2$

& consider the inclusion

$$K(n)_* \hookrightarrow (K_n)_*$$

$$v_n \longmapsto u^{1-p^n}$$

- This is a flat extension

$$\underbrace{(K_n)_*(X) = K(n)_*(X) \otimes_{K(n)_*} (K_n)_*}$$

is a homology theory $\rightarrow K_n$

$\Longrightarrow$ K_n a 2-periodic Morava K-Theory with FGL F_{K_n}.

Using u we can actually shift F_{K_n} to a non-graded FGL / $(K_n)_* = \mathbb{F}_{p^n}$

$$\Gamma_n = u^{-1} F_{K_n}(ux, uy)$$

its p-series $\quad [p]_{\Gamma_n}(x) = x^{p^n}$

it is non-graded Honda FGL / $\mathbb{F}_{p^n}$

We can do similar operation on Morava E-theory.

take $\quad F_{E_n} = u^{-1} F(ux, uy)$

$$[p]_{F_{E_n}} = px +_{F_{E_n}} u_1 x^p +_{F_{E_n}} \cdots +_{F_{E_n}} x^{p^n}$$

this is a non-graded FGL / $(E_n)_0 = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$

$$\begin{array}{ccc} F_{E_n} & (E_n)_0 \simeq & \underbrace{W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]}_{\text{Complete local ring}} \\ \downarrow & \Big\downarrow \pi \; (p, u_1, \dots, u_{n-1}) = m & \\ \Gamma & \mathbb{F}_{p^n} & \end{array}$$

in

Hence F_{E_n} is a deformation of Γ_n

Lubin-Tate tells us $((E_n)_0, F_{E_n})$ is the universal deformation of $(\Gamma_n, \mathbb{F}_{p^n})$

$$
\begin{tikzcd}
F_{E_n} \arrow[d] & (E_n)_0 \arrow[r, dashed, "\exists!"] \arrow[d, "\pi"'] & B \arrow[d, "\pi"] & G \arrow[d] \\
\Gamma_n & \mathbb{F}_{p^n} \arrow[r, "i", "\simeq"'] & B/m & \pi^*(G) \cong i_*(\Gamma_n)
\end{tikzcd}
$$

B : complete local ring with maximal ideal m

- $\underset{\mathbb{S}_n}{\overset{\parallel}{\mathrm{Aut}(\Gamma_n)}}$ acts on $(E_n)_0$

small morava stabilizer grp

$g \in \mathrm{Aut}(\Gamma_n)$

$$
\begin{tikzcd}
(E_n)_0 \arrow[r, dashed, "\tilde{g}"] \arrow[d] & (E_n)_0 \arrow[d, "\pi"] & g^{-1} F_E(g(x), g(y)) \arrow[d] \\
\mathbb{F}_{p^n} \arrow[r, "id"'] & \mathbb{F}_{p^n} & \Gamma_n
\end{tikzcd}
$$

- $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $(E_n)_0$

$$\begin{tikzcd}
(E_n)_0 \arrow[r, dashed, "\tilde{\sigma}"] \arrow[d] & (E_n)_0 \arrow[d] \arrow[d, bend left=60, "f_E"] \\
\mathbb{F}_{p^n} \arrow[r, "\sigma"] & \mathbb{F}_{p^n} \\
\Gamma_n \arrow[r, squiggly] & \Gamma_n
\end{tikzcd}$$

$\sigma \in \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$

Hence $(E_n)_*$ admits $\mathbb{G}_n = \mathrm{Aut}(\Gamma_n) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ action.

(big) Morava stabilizer group.

$\mathbb{G}_n \curvearrowright (E_n)_*$ · profinite p-group.

Thm (Goerss–Hopkins–Miller)

The $\mathbb{G}_n$-action on $(E_n)_*$ can be lifted uniquely to an action of $\mathbb{G}_n$ on E_n via $\mathbb{E}_\infty$-ring maps.

Thm (Devinatz-Hopkins)

$$E_n^{h\mathbb{G}_n} \simeq \underline{L_{K(n)} S^0}$$

$\hookrightarrow L_{K(n)}$: Bousfield localization

- E_1 is $KU_p^\wedge$ $\quad \mathbb{G}_1 = \mathbb{Z}_p^\times$

where $\mathbb{Z}_p^\times \curvearrowright E_1$ by Adams operations.

- when $p=2$ $\quad C_2 \subseteq \mathbb{Z}_2^\times$

$$(KU_2^\wedge)^{hC_2} = KO_2^\wedge$$

2-complete real K-theory

- $G \subseteq \mathbb{G}_n$ finite grp

E_n^{hG} higher real K-theory.