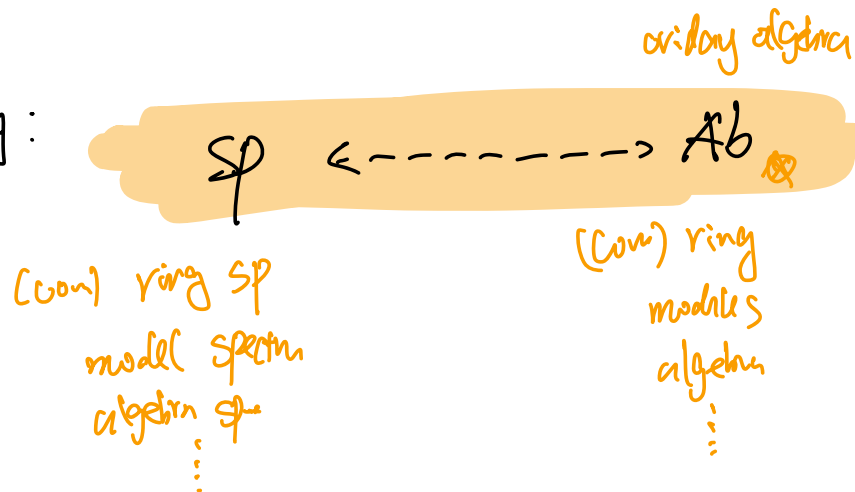


Morava K & E Theories

invar 2022

§0 Some higher algebra

Analogy:



We need a rigid model for Sp , not just $Ho(Sp)$

• loose many info

- EKMM S -modules
 - Sym/orth spectra
 - ∞ -cat of Sp .
- } $\rightsquigarrow Ho(Sp)$

• a sym monoidal cat of spectra: $Sp \wedge$

Def: • A ring spectrum is a monoid in $\mathcal{S}p$

• A commutative ring spectrum is a commutative monoid in $\mathcal{S}p$.

• we have similar notions in $(\mathcal{H}o\mathcal{I}S\mathcal{P})$

• In point-set level, we have different operads $(E_n)_{n \geq 1}$

to encode higher homotopy coherently

ring spectra $\longleftrightarrow A_{\infty}''$ -spectra

com ring spectra $\longleftrightarrow E_{\infty}$ -spectra

Def: Let R be a ring spectrum, a (left) R -module

is a spectrum M with $\varphi: R \wedge M \rightarrow M$ s.t.

$$S \wedge M \xrightarrow{\text{id}} R \wedge M \quad R \wedge R \wedge M \xrightarrow{\text{assoc}} R \wedge M$$

$$\begin{array}{ccc} \downarrow \cong & \dots & \downarrow \varphi \\ & \searrow & \\ & & M \end{array} \quad \begin{array}{ccc} \downarrow \text{id} \wedge \varphi & & \downarrow \varphi \\ R \wedge M & \xrightarrow{\varphi} & M \end{array}$$

• \mathcal{M}_R : cat of R -modules, complete & complete

let R be a com ring spectrum, \mathcal{D}_R be the derived category of \mathcal{M}_R

Def: An R -ring spectrum A is an R -module A with unit $\eta: R \rightarrow A$ & product $\phi: A \wedge_R A \rightarrow A$ s.t. the following diagram commutes in \mathcal{D}_R

$$\begin{array}{ccc}
 R \wedge_R A & \longrightarrow & A \wedge_R A & \longleftarrow & A \wedge_R R \\
 & \searrow & \downarrow \phi & \swarrow & \\
 & \dots & A & \dots &
 \end{array}$$

A is ass or com if the appropriate diagrams commute

- \mathcal{A} should understand as derived Λ .

prop: • If A & B are R -ring spectra, then so is

$A \wedge_R B$.

See modern foundation of stable homotopy theory

- under some mild conditions, cofibers A/x $\in \text{fmm}$

localizations $R[y^{\pm 1}]$ both admit R -ring structures

- $M(k) = \text{cofib}(\sum^{2(p(k)-1)} BP \xrightarrow{\cdot v_k} BP)$

- $BP[v_n^{\pm 1}]$

Both are BP -ring spectra.

§ I Landweber exact functor thm

$$\begin{array}{ccc} \text{CoCT} & \longrightarrow & \text{FGL} \\ (E, \chi^E) & \longmapsto & F \in E_*[[x, y]] \end{array}$$

A natural Question:

$$f: MU_* \longrightarrow R_*$$

Given a formal group law F/R_* , can we find a spectrum $s.t$ its fgl is F .

A possible candidate: $MU_* \otimes_{MU_*} R_*$

If it is a homology theory

$$\begin{array}{ccc} MU_* & \longrightarrow & MU_* \otimes_{MU_*} R_* \\ \downarrow \simeq & & \\ f: MU_* & \longrightarrow & R_* \end{array}$$

- However, it is not a homology theory in general

- Homotopy axiom
- Suspension axiom
- Additivity axiom

• **exactness**

Given a cofiber sequence $X \rightarrow Y \rightarrow Z$ in CW-complexes

$$MU_*(X) \otimes_{MU_*} R_* \rightarrow MU_*(Y) \otimes_{MU_*} R_* \rightarrow MU_*(Z) \otimes_{MU_*} R_*$$

might not be exact in general

$$\begin{array}{ccc} MU_*\text{-mod} & \longrightarrow & Ab \\ M & \longrightarrow & M \otimes_{MU_*} R_* \end{array} \quad - \otimes_{MU_*} R_*$$

When is it exact??

- If R_* is a flat MU_* -module

$$\bullet MU_* = \pi_*(MU) = \mathbb{Z}\langle x_1, x_2, \dots \rangle$$

flat modules / MU_* are too rare.

- $MU_*(\mathbb{X})$ enjoys a richer structure

Recall: (MU_*, MU_*MU) is a Hopf algebra.

Def: A (MU_*, MU_*MU) -comodule M is a MU_* -module

together with a co-action map

$$M \longrightarrow MU_*MU \otimes_{MU_*} M$$

which is compatible with the multiplication on MU_*MU .

- $MU_*(\mathbb{X})$ is a (MU_*, MU_*MU) -comodule

the co-action map is induced

$$\begin{array}{ccc} MU_* \wedge \mathbb{X} & \xrightarrow{id \wedge id} & MU_* MU_* \wedge \mathbb{X} \\ \parallel & & \parallel \\ MU_* \Sigma^0 \wedge \mathbb{X} & & \end{array}$$

Consider the functor

$$\begin{array}{ccc} (\mathcal{M}_{\mathcal{U}_*}, \mathcal{M}_{\mathcal{U}_*} \mathcal{M}_{\mathcal{U}}) \text{-comod} & \longrightarrow & \mathcal{A}b \\ M & \longmapsto & M \otimes_{\mathcal{M}_{\mathcal{U}_*}} \mathcal{R}_* \end{array}$$

When is this functor exact??

Recall:

Def: Given a FGL F/\mathcal{R}_* , let v_i be the

coefficient of x^i in $[p]_{\mathcal{F}}(x)$.

$$= x^{\dagger_{\mathcal{F}}} x_{\mathcal{F}}^{\dagger} \cdots \dagger_{\mathcal{F}} x$$

Thm (Landweber)

Let \mathcal{R}_* be an $\mathcal{M}_{\mathcal{U}_*}$ -algebra. Then the functor

$$\begin{array}{ccc} (\mathcal{M}_{\mathcal{U}_*}, \mathcal{M}_{\mathcal{U}_*} \mathcal{M}_{\mathcal{U}}) \text{-comod} & \longrightarrow & \mathcal{A}b \\ M & \longmapsto & M \otimes_{\mathcal{M}_{\mathcal{U}_*}} \mathcal{R}_* \end{array}$$

is exact iff for any $p \in n$, the map

$$R_*/(p, v_1, \dots, v_{n-1}) \xrightarrow{\cdot v_n} R_*/(p, v_1, \dots, v_n)$$

is injective. (p, v_1, \dots, v_n) is a regular sequence in R_* .

In particular, $MU_* \otimes_{MU_0} R_*$ is a homology theory.

• If a coct E satisfies this condition

Then we say E is a Landweber theory

p -typical FGL version. $\longleftrightarrow BP_*$

replace 1) FGL by p -typical FGL

2) MU_* by BP_*

(BP_0, BP_1, BP) also a Hopf algebraic

$$(BP_*, BP_1, BP) \text{-comod} \longrightarrow Ab$$

is exact iff $\forall n$

$$R_*/(p, v_1, \dots, v_{n-1}) \xrightarrow{\cdot v_n} R_*/(p, v_1, \dots, v_n)$$

is injective

Ex:

1) $H\mathbb{Z}$ with additive FGL: $F(x,y) = x+y$

$$[p]_{\mathbb{F}}(x) = px \quad \rightarrow \quad \begin{cases} v_0 = p \\ v_i = 0 \quad i > 0 \end{cases}$$

$$\bullet \quad \mathbb{Z}/(p) \xrightarrow{\cdot v_i} \mathbb{Z}/(p)$$

is not injective!!

• $H\mathbb{Z}$ is not landweher

• $H\mathbb{Q}$ is landweher: Height 0 theory

2) Complex k -Theory KU with multiplication FGL

$$F = x+y + \beta xy \quad KU_* \simeq \mathbb{Z}[\beta^{\pm 1}]$$

$$\begin{aligned} [p]_{\mathbb{F}}(x) &= \beta^{-1} ((\beta x + 1)^p - 1) \\ &= px + \dots + \beta^{p-1} x^{p-1} \end{aligned} \quad \rightarrow \quad \begin{cases} v_0 = p \\ v_i = \beta^{p-i} \\ \dots \end{cases}$$

$v_i = 0 \quad \perp$

$$\begin{array}{ccc} \bullet & K\mathbb{A}_* / (c_p) & \xrightarrow{\cdot v_1} & K\mathbb{A}_* / (c_p) \\ & \text{SI} & & \\ & \mathbb{F}_p[\beta^{\neq 1}] & \xrightarrow{\cdot \beta^{p-1}} & \mathbb{F}_p[\beta^{\neq 1}] \end{array}$$

$$\begin{array}{ccc} \bullet & K\mathbb{A}_* / (c_p, v_1) & \xrightarrow{\cdot v_2} & K\mathbb{A}_* / (c_p, v_1) \\ & \parallel & & \parallel \\ & 0 & & 0 \end{array}$$

$\Rightarrow (K\mathbb{A}_*, \mathbb{F}_*)$ satisfies the Landweber exactness

$$\Rightarrow K\mathbb{A}_*(X) \simeq M\mathbb{A}_*(X) \otimes_{M\mathbb{A}_*} K\mathbb{A}_*$$

Conner-Floryd thm 1966

3) Johnson-Wilson theory $E(n)$

$$\text{Let } E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n^{\#}]$$

Consider the natural map

gives a p -typical
FGL / $E(n)_*$.

$$BP_* \longrightarrow E(n)_*$$

$$v_i \longmapsto \begin{cases} v_i & i \leq n \\ 0 & i > n \end{cases}$$

$$\bullet \quad E(n)_* / (p, v_1, \dots, v_i) \xrightarrow{-v_{i+1}} E(n)_* / (p, v_1, \dots, v_i)$$

$$i < n \quad \mathbb{F}_p[v_{i+1}, \dots, v_n^{\#}] \xrightarrow{-v_{i+1}} \mathbb{F}_p[v_{i+1}, \dots, v_n^{\#}]$$

$$i \geq n \quad 0 \longrightarrow 0$$

So Landweber gives a spectrum $E(n)$ which is
height $\leq n$

§ 2 Morava K-Theory.

Does it exist a height exactly n complex oriented spectrum?

A naive idea:

$$K(n)_* = \mathbb{F}_p [v_n^{\pm 1}]$$

with $BP_* \longrightarrow K(n)_*$

$$\begin{cases} v_n & \longrightarrow v_n \\ \text{others} & \longrightarrow 0 \end{cases}$$

However, we cannot construct such spectrum from

Landweber exactness.

$$\begin{array}{ccc} K(n)_*/(p) & \xrightarrow{\cdot v_1} & K(n)_*/(p) \\ \uparrow \cong & & \uparrow \cong \\ \mathbb{F}_p & & \mathbb{F}_p \end{array}$$

if $n \geq 2$ $v_1 = 0$

is not inject.

... $n=1$

Def: $K(n) := \text{BP} \langle v_n^{\#} \rangle \wedge_{\text{BP}} \bigwedge_{k \neq n} M(k)$ in \mathcal{D}_{BP}

- $K(n)_* = \mathbb{F}_p \langle v_n^{\#} \rangle$

- $K(n)$ is a homotopy ass ring spectrum

- If $p > 2$ $K(n)$ a homotopy commutative ring spectrum

!! Not a E_{∞} -ring spectrum

- $\text{BP}_* \rightarrow K(n)_*$ classifies a height exactly n p -typical FGL. $\mathbb{F} \langle v_n^{\#} \rangle \cong \mathbb{F}_p[x] \cong \mathbb{F}_p[x^{p^n}]$
Hondal's FGL

- $K(n)$ is a summand of mod p complex K-theory ku/p .

prop: • Künneth thm

$$K(n)_* (X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$$

• let X be a p -local finite CW-complex

$$\text{Then } K(n)_*(X) = 0 \implies K(n-1)_*(X) = 0$$

Thick subcat thm.

§ 3: Morava E-Theory

$$\text{Set } (E_n)_* = W(\mathbb{F}_p^n) [[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$$

• $W(\mathbb{F}_p^n)$: Witt vectors of \mathbb{F}_p^n
complete local ring with maximal ideal (p)

$$W(\mathbb{F}_p^n) / (p) = \mathbb{F}_p^n$$

$$\bullet |u_i| = 0 \quad |u| = -2$$

$$\text{Consider } BP_* \longrightarrow (E_n)_*$$

$\circ \dots \circ \mathbb{F}_p^n$

$$U_i \mapsto \begin{cases} u_i u^i & 0 \leq i < n \\ u^{1-p^n} & i = n \\ 0 & i > n \end{cases}$$

it defines a p -typical FGL $\mathbb{F}/(E_n)_*$ with

$$\Gamma(p)_{\mathbb{F}}(x) = px + \sum_{\mathbb{F}} u_i u^{ip^i} x^{p^i}$$

• Since u is invertible, the sequence

$(p, u, u^{1+p}, u^2 u^{1+p^2}, \dots, u^{1+p^n}, 0, \dots)$ satisfies

the condition in Landweber

\Rightarrow a complex oriented spectrum E_n

called Morava E -Theory.

Why $(E_n)_*$ looks like this ??

Motivation comes from Deformation theory.

... $(E_n)_*$

2-periodic Morava K-theory:

$$K(n)_* \cong \mathbb{Z}[v_n] \quad |v_n| = 2(p^n - 1)$$

$$(K_n)_* = \mathbb{F}_p[u^{\pm 1}] \quad |u| = -2$$

define $(K_n)_* = \mathbb{F}_p[u^{\pm 1}]$ $|u| = -2$ $|u| = -2$

ring

& consider the inclusion

$$\begin{array}{ccc} K(n)_* & \hookrightarrow & (K_n)_* \\ v_n & \longmapsto & u^{p^n} \end{array}$$

- This is a flat extension

$$(K_n)_*(X) = K(n)_*(X) \otimes_{K(n)_*} (K_n)_*$$

is a homology theory $\Rightarrow K_n$

$\Rightarrow K_n$ a 2-periodic Morava K-theory

with FGL F_{K_n} .

Using u we can actually shift F_{K_n} to
a non-graded FGL $\mathbb{F}_p[u^{\pm 1}] = \mathbb{F}_p$

$$\Gamma_n = \mathcal{U}^\dagger \mathbb{F}_{\mathbb{F}_n}(u_x, u_y)$$

its p -series $[p]_{\mathbb{F}_n}(x) = x^{p^n}$

it is non-graded Hopf algebra / \mathbb{F}_n

We can do similar operation on Morava E -theory.

take $\mathbb{F}_E = \mathcal{U}^\dagger \mathbb{F}(u_x, u_y)$

$$[p]_{\mathbb{F}_E} = px + \frac{u_x x^p}{\mathbb{F}_E} + \dots + \frac{x^{p^n}}{\mathbb{F}_E}$$

this is a non-graded FGL / $(E_n)_0 \cong W(\mathbb{F}_p)[[u_1, \dots, u_{n-1}]]$

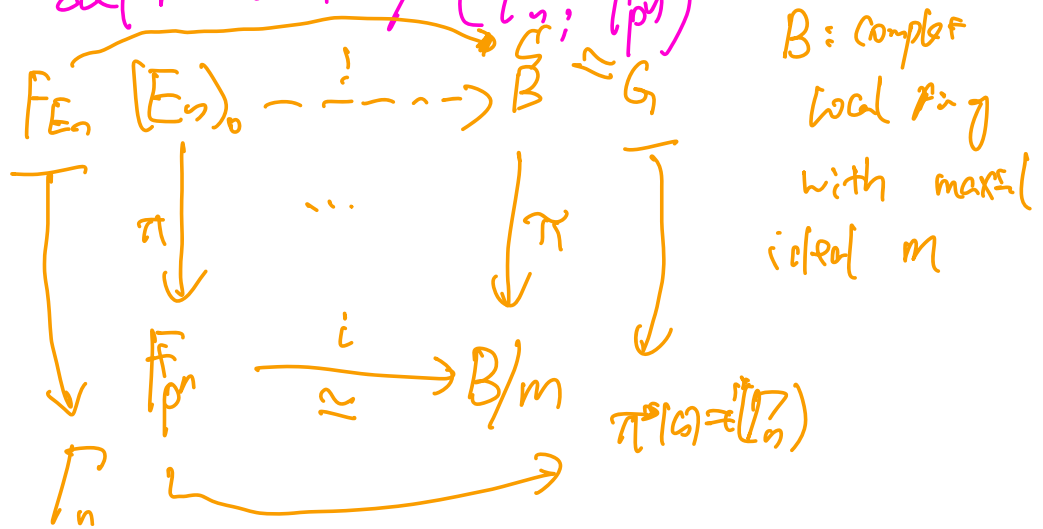
$$\begin{array}{ccc}
 \mathbb{F}_E & (E_n)_0 \cong W(\mathbb{F}_p)[[u_1, \dots, u_{n-1}]] & \\
 \downarrow & \downarrow \pi (p, u_1, \dots, u_{n-1}) \cong m & \text{Complete local reg} \\
 \mathbb{F} & \mathbb{F}_p &
 \end{array}$$

in 1

Hence F_{E_n} is a deformation of Γ_n

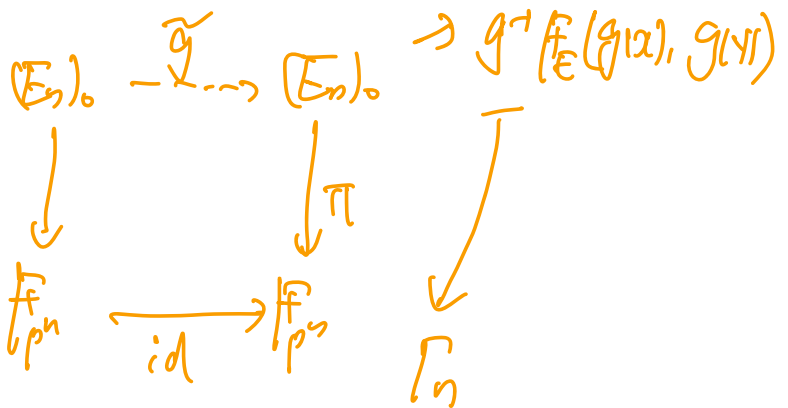
Lubin-Tate tells us $(E_n)_0; F_{E_n}$ is the

universal deformation / $(\Gamma_n; \mathbb{F}_p)$

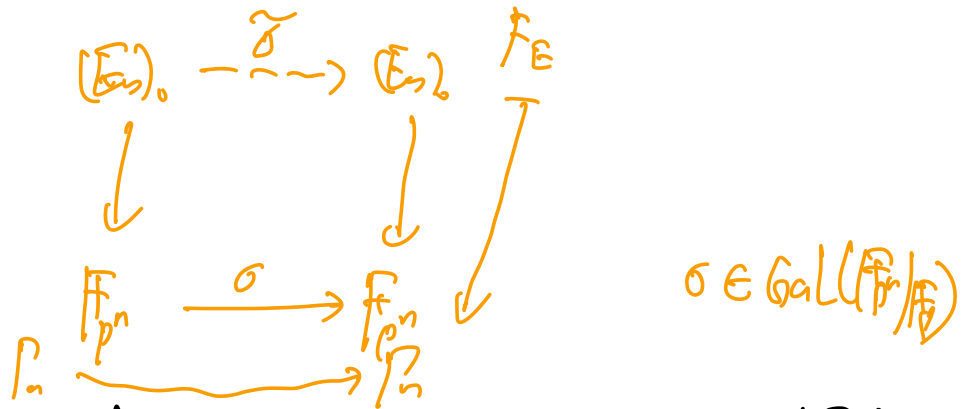


• $\text{Aut}(\Gamma_n)$ acts on $(E_n)_0$ $g \in \text{Aut}(\Gamma_n)$

Small morava
stabilizer group



- $\text{Gal}(F_p^n/F_p)$ acts on $(E_p)_0$.



Hence $(E_p)_*$ admits $G_n = \text{Aut}(P_n) \rtimes \text{Gal}(F_p^n/F_p)$

(big) Morava stabilize group.

action.

$G_n \curvearrowright (E_p)_*$ • profinite p -group.

Thm (Gøeravss - Hopkins - Miller)

The G_n -action on $(E_p)_*$ can be lifted

uniquely to an action of G_n on E_p via

E_∞ -ring maps.

Thm (Devnatz-Hopkins)

$$E_n^{hG_n} \simeq \underline{L_{K(n)} S^0}$$

$\hookrightarrow L_{K(n)}$: Bousfield localization

- E_1 is KU_p $G_1 = \mathbb{Z}_p^\times$

where $\mathbb{Z}_p^\times \curvearrowright E_1$ by Adams operations.

- when $p=2$ $C_2 \subseteq \mathbb{Z}_2^\times$

$$(KU_2^\wedge)^{hC_2} = KO_2^\wedge$$

2-complete real K-theory

- $G \subseteq G_n$ finite group

E_n^{hG} higher real K-theory.