Thick Subcategories and Balmer Spectrum
IWoAT 2022

Weinan Lin

Peking University

Aug 22 2022

Type n

Recall that for each n we have Morava K-theory $K(n)$ with

$$\pi_*(K(n)) \cong \mathbb{F}_p[v_n^{\pm 1}].$$

It has a height n (exactly) p-typical formal group law

$$[p]_{F_{K(n)}}(x) = v_n x^{p^n}.$$

Type n

Recall that for each n we have Morava K-theory $K(n)$ with

$$\pi_*(K(n)) \cong \mathbb{F}_p[v_n^{\pm 1}].$$

It has a height n (exactly) p-typical formal group law

$$[p]_{F_{K(n)}}(x) = v_n x^{p^n}.$$

Lemma

Let X be a finite *p-local* spectrum. Then $K(n)_*(X) \cong 0$ implies $K(n-1)_*(X) \cong 0$ for $n > 0$.

Type n

Definition

We say that a p-local finite spectrum X has type n if $K(n)_*(X) \neq 0$ but $K(m)_*(X) \cong 0$ for $m < n$.

For example, X has type 0 if $H_*(X; \mathbb{Q}) \neq 0$, or equivalently if $H_*(X; \mathbb{Z})$ is not a torsion group.

Type n

Definition

We say that a p-local finite spectrum X has type n if $K(n)_*(X) \neq 0$ but $K(m)_*(X) \cong 0$ for $m < n$.

For example, X has type 0 if $H_*(X; \mathbb{Q}) \neq 0$, or equivalently if $H_*(X; \mathbb{Z})$ is not a torsion group.

Remark

Let X be a finite p-local spectrum. Then $H_*(X; \mathbb{F}_p) \cong 0$ if and only if $X \simeq 0$. Moreover, $H_k(X; \mathbb{F}_p)$ vanishes for almost all values of K. For sufficiently large n, the Atiyah-Hirzebruch spectral sequence for $K(n)_*(X)$ degenerates to give $K(n)_*(X) \cong H_*(X; \mathbb{F}_p)[v_n^{\pm 1}]$. It follows that if $X \not\simeq 0$, then $K(n)_*(X) \neq 0$ for sufficiently large n.

Every nonzero finite p-local spectrum X has type n for some unique n. By convention, we will say that the spectrum 0 has type ∞.

Thick Subcategories

If we consider finite p-local spectra of type $\geq n$, we can find several formal properties of this collection. We make the following formal definition.

Definition

Let $\mathcal{T}$ be a full subcategory of finite p-local spectra. We say that $\mathcal{T}$ is *thick* if it contains 0, is closed under the formation of fibers and cofibers, and if every retract of a spectrum belonging to $\mathcal{T}$ also belongs to $\mathcal{T}$.

Proposition

Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. If $X \in \mathcal{T}$ and Y is any finite p-local spectrum, then $X \otimes Y \in \mathcal{T}$.

Indeed, the collection of p-local finite spectra Y for which $X \otimes Y \in \mathcal{T}$ is itself thick. Since it contains the p-local sphere $S_{(p)}$, it contains all finite p-local spectra. (every finite p-local spectrum admits a finite cell decomposition.)

Thick Subcategories

Definition

Let $\mathcal{C}_{\geq n}$ be the collection of finite p-local spectra which have type $\geq n$. In other words, $X \in \mathcal{C}_{\geq n}$ if and only if $K(m)_*(X) \cong 0$ for $m < n$.

Using the long exact sequence in $K(m)$-homology, we see that if we are given a cofiber sequence

$$X' \to X \to X'',$$

and any two of X', X and X'' have type $\geq n$, then so does the third. Moreover, it is clear that any retract of a spectrum of type $\geq n$ is also of type $\geq n$. Consequently,

Proposition

$\mathcal{C}_{\geq n}$ is a thick subcategory of the category of finite p-local spectra.

Thick Subcategories

Question

What do all thick subcategories of finite p-local spectra look like?

Thick Subcategories

Question

What do all thick subcategories of finite p-local spectra look like?

Thick Subcategory Theorem

Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. Then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $0 \leq n \leq \infty$.

A sketch of the proof will be given in tomorrow's talk.

Balmer spectrum

If we put all the thick subcategories $\mathcal{C}_{\geq n}$ of finite p-local spectra for all $n \geq 1$ and all prime p together, we can get something called the Balmer spectrum. More precisely,

Definition

Let $(\mathcal{T}, \otimes, \mathbf{1})$ be an *essentially small $\otimes$-triangulated category*, the Balmer spectrum $\mathrm{Spc}(\mathcal{T})$ is a space whose points are the prime thick $\otimes$-ideals, i.e., the proper thick $\otimes$-ideals $\mathcal{I} \subset \mathcal{T}$ such that if $a \otimes b \in \mathcal{I}$, then $a \in \mathcal{I}$ or $b \in \mathcal{I}$. A basis for the open subsets of $\mathrm{Spc}(\mathcal{T})$ is given by the complements of subsets of the form $\mathrm{Supp}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid a \notin \mathcal{P}\}$.

Balmer spectrum

If we put all the thick subcategories $\mathcal{C}_{\geq n}$ of finite p-local spectra for all $n \geq 1$ and all prime p together, we can get something called the Balmer spectrum. More precisely,

Definition

Let $(\mathcal{T}, \otimes, \mathbf{1})$ be an *essentially small* $\otimes$*-triangulated category*, the Balmer spectrum $\mathrm{Spc}(\mathcal{T})$ is a space whose points are the prime thick $\otimes$-ideals, i.e., the proper thick $\otimes$-ideals $\mathcal{I} \subset \mathcal{T}$ such that if $a \otimes b \in \mathcal{I}$, then $a \in \mathcal{I}$ or $b \in \mathcal{I}$. A basis for the open subsets of $\mathrm{Spc}(\mathcal{T})$ is given by the complements of subsets of the form $\mathrm{Supp}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid a \notin \mathcal{P}\}$.

Consider the category of finite spectra Sp^{ω}, by the Thick Subcategory Theorem, we have

$$\mathrm{Spc}(Sp^{\omega}) = \{\mathcal{C}_{\geq n,p} \mid 1 \leq n \leq \infty,\ p \text{ any prime}\}.$$

with topology given in the above definition.

Question

What about other categories other than Sp^{ω}?

It would be interesting to consider the Balmer spectrum of the category Sp_G^{ω} of *compact genuine G-spectra* ([LMS]) for a finite group G.
For each subgroup H of G we have an exact symmetric monoidal geometric fixed point functor

$$\Phi^H : Sp_G^{\omega} \to Sp^{\omega}$$

which induces a continuous map of Blamer spectra

$$\Phi^{H^*} : \mathrm{Spc}(Sp^{\omega}) \to \mathrm{Spc}(Sp_G^{\omega}).$$

Balmer spectrum

$$\Phi^{H^*} : \mathrm{Spc}(Sp^\omega) \to \mathrm{Spc}(Sp^\omega_G).$$

Balmer and Sanders show that these maps are *jointly surjective* and $\Phi^{H_1^*}(\mathfrak{p}) = \Phi^{H_2^*}(\mathfrak{q})$ if and only if H_1 is conjugate to H_2 and $\mathfrak{p} = \mathfrak{q}$. This identifies $\mathrm{Spc}(Sp^\omega_G)$ as a set.
To identify the topology on $\mathrm{Spc}(Sp^\omega_G)$ one needs to further identify all the inclusions between the prime ideals

$$\left\{\mathcal{P}(H, p, n) := (\Phi^H)^{-1}(\mathcal{C}_{\geq n,p}) \mid H \subset G, 1 \leq n \leq \infty, p \text{ prime}\right\}$$

Balmer spectrum

Theorem (BHNNNS)

When G is an abelian group, $K \subset H \subset G$ are subgroups, p is a prime and $1 \leq n < \infty$, then the minimal i such that

$$\mathcal{P}(K, p, n) \subset \mathcal{P}(H, p, n - i)$$

is $i = \mathrm{rank_p}(H/K)$.

The p-rank here of an abelian group A is $\dim_{\mathbb{F}_p}(A/p)$. For example, the 2-rank of $A = Z/2 \oplus Z/8 \oplus Z/16$ is 3.

References

Lurie's lecture notes:
https://www.math.ias.edu/~lurie/252xnotes/Lecture26.pdf

Ravenel's Orange book:
https://people.math.rochester.edu/faculty/doug/nilp.html

Balmer spectrum:
http://www.ms.uky.edu/~njst237/papers/sixauthors.pdf

IWoAT2019 Summer school on equivariant homotopy theory
https://iwoat.github.io/2019/school

Thank You!