Nilpotence theorem
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Definition

We say that a ring spectrum E is a field if $\pi_* E$ is a graded field.

Every Morava K-theory is a field.
Conversely, if E is any field, then we claim that E has the structure of a $K(n)$-module for some $0 \leq n \leq \infty$. It suffices to show that $E \otimes K(n)$ is nonzero for some n because of the following.

Proposition

Let E be any field and suppose that $E \otimes K(n)$ is nonzero. Then E admits the structure of a $K(n)$-module.

Nilpotence theorem

Proposition 1

Let $\{E^\alpha\}$ be a collection of ring spectra. The following conditions are equivalent:

1. Let R be a p-local ring spectrum. If $x \in \pi_m R$ is a homotopy class whose image in $E^\alpha_m(R)$ is zero for all α, then x is nilpotent in $\pi_* R$.
2. Let R be a p-local ring spectrum. If $x \in \pi_0 R$ is a homotopy class whose image in $E^\alpha_0(R)$ is zero for all α, then x is nilpotent in $\pi_0 R$.
3. Let X be an arbitrary p-local spectrum. If $x \in \pi_0 X$ has trivial image under the Hurewicz map $\pi_0 X \to E^\alpha_0(X)$ for each α, then the induced class $x^{\otimes n} \in \pi_0 X^{\otimes n}$ is zero for n sufficiently large.
4. Let X be an arbitrary p-local spectrum, and let F be a finite spectrum. If $f : F \to X$ is such that each composite map $F \to X \to X \otimes E^\alpha_0$ is null homotopic, then $f^{\otimes n} : F^{\otimes n} \to X^{\otimes n}$ is nullhomotopic for n sufficiently large.

Nilpotence theorem

Nilpotence Theorem (Devinatz, Hopkins, and Smith)

For any ring spectrum R, the kernel of the map $\pi_* R \to MU_*(R)$ consists of nilpotent elements. In particular, the single cohomology theory MU detects nilpotence.

Corollary (Nishida)

For $n > 0$, every element of $\pi_n S$ is nilpotent.

This is because $\pi_n S$ is torsion and $MU_*(S) = L$ is torsion free.

Nilpotence theorem

Theorem 1

The spectra $\{K(n)\}_{0 \leq n \leq \infty}$ detect nilpotence.

We will prove that the spectra $\{K(n)\}_{0 \leq n \leq \infty}$ satisfy condition (3) of Proposition 1. Let T denote the homotopy colimit of the spectra

$$S \xrightarrow{x} X \xrightarrow{x} X^{\otimes 2} \xrightarrow{x} X^{\otimes 3} \xrightarrow{x} \cdots .$$

Nilpotence theorem

Let $x \in \pi_0 X$ and E be any ring spectrum.

Lemma

The following conditions are equivalent

1. The spectrum T is E-acyclic.
2. The image of $x^{\otimes n}$ in $E_0(X^{\otimes n})$ vanishes when n is sufficiently large.

Nilpotence theorem

We now want to prove Theorem 1. Assume the image of $x \in \pi_0 X$ in each $K(n)_0 X$ is zero. We wish to prove that some smash power $x^{\otimes n}$ is trivial.

Nilpotence theorem

Since $K(m)$ satisfy the Künneth theorem, $x^{\otimes n}$ has trivial image in $K(m)_*(X^{\otimes n})$ if and only if x has trivial image in $K(m)_*(X)$.
Consequently, we have a more precise result for a homotopy class $x \in \pi_0 X$ for a p-local spectrum X:

Proposition

the $x^{\otimes n} \in \pi_0 X^{\otimes n}$ is zero for n sufficiently large *if and only if* the image of x in $K(m)_*(X)$ vanishes for all m.

Remark

We can drop the requirement that X is p-local if we impose the same condition at all Morava K-theories (for all primes).

Nilpotence theorem

Now we have proved our claim:

Corollary

If E is a field, then E has the structure of a $K(n)$-module for some n.

Thick Subcategory Theorem

Definition

Let $\mathcal{T}$ be a full subcategory of finite p-local spectra. We say that $\mathcal{T}$ is *thick* if it contains 0, is closed under the formation of fibers and cofibers, and if every retract of a spectrum belonging to $\mathcal{T}$ also belongs to $\mathcal{T}$.

Proposition

Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. If $X \in \mathcal{T}$ and Y is any finite p-local spectrum, then $X \otimes Y \in \mathcal{T}$.

Thick Subcategory Theorem

Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. Then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $0 \leq n \leq \infty$.

Thick Subcategory Theorem

The Thick Subcategory Theorem is equivalent to the following proposition.

Proposition

Let $\mathcal{T}$ be a thick subcategory containing a type n spectrum X. If Y is a spectrum of type $\geq n$, then $Y \in \mathcal{T}$.

Let DX denote the (p-local) Spanier-Whitehead dual of X. Then the map $e : S_{(p)} \to X \otimes DX$ induces an injection

$$K(m)_*(S_{(p)}) \to K(m)_*(X \otimes DX) \cong K(m)_*(X) \otimes_{\mathbb{F}_p[v_m^{\pm 1}]} K(m)_*(X)^\vee$$

Thick Subcategory Theorem

Consider the fiber sequence

$$F \xrightarrow{f} S_{(p)} \to X \otimes DX.$$

If follows that the map $K(m)_*F \to K(m)_*(S_{(p)})$ is zero for $m \geq n$. Consider the composite

$$g : F \xrightarrow{f} S_{(p)} \to Y \otimes DY$$

Then $g_* : K(m)_*F \to K(m)_*(Y \otimes DY)$ is trivial for both $m \geq n$ and $m < n$ (because Y has type $\geq n$, so that $K(m)_*(Y \otimes DY) \cong 0$.)

Thick Subcategory Theorem

By the nilpotence theorem, we conclude that some smash power

$$F^{\otimes k} \to (Y \otimes DY)^{\otimes k}.$$

is null homotopic. Composing with the multiplication on $Y \otimes DY$ we get a nullhomotopic map

$$F^{\otimes k} \to Y \otimes DY.$$

which correspond to the composition

$$F^{\otimes k} \otimes Y \xrightarrow{f} F^{\otimes k-1} \otimes Y \xrightarrow{f} \cdots \to Y.$$

Thick Subcategory Theorem

It follows that Y is a retract of the cofiber $Y/(F^{\otimes k} \otimes Y)$. Consequently, to show that $Y \in \mathcal{T}$, it suffices to show that $Y/(F^{\otimes k} \otimes Y) \in \mathcal{T}$.
Since $\mathcal{T}$ is closed in the formation of cofiber and fiber, it suffices to show that $(F^{\otimes a} \otimes Y)/(F^{\otimes a+1} \otimes Y) \in \mathcal{T}$ for $a = 0, 1, \ldots, k-1$.
In fact, each of them has the form

$$F^{\otimes a} \otimes Y \otimes (S_(p)/F) \simeq F^{\otimes a} \otimes Y \otimes DX \otimes X$$

and therefore belong to $\mathcal{T}$ since $X \in \mathcal{T}$.

References

Lurie's lecture notes:

https://www.math.ias.edu/~lurie/252xnotes/Lecture26.pdf

https://www.math.ias.edu/~lurie/252xnotes/Lecture27.pdf

Thank You!