Construction of v_n-self maps

Yutao Liu

University of Chicago

IWOAT, 2022.8

References

Jacob Lurie's notes on chromatic homotopy theory (lecture 27): https://people.math.harvard.edu/∼lurie/252x.html

D. C. Ravenel: "Nilpotence and Periodicity in Stable Homotopy Theory", chapters 6, B, C.

D. C. Ravenel: "Complex Cobordism and Stable Homotopy Groups of Spheres", chapters 2, 4.

Introduction

- Everything is p-local in this talk. $H^*(-)$ will denote $H^*(-;\mathbb{Z}/p)$.
- Morava K-theory $K(n)$: $K(n)_* = \mathbb{Z}/p[v_n^{\pm}]$, $|v_n| = 2p^n - 2$.
- A finite spectrum X has **type n** if n is the smallest integer such that $\overline{K(n)}_*(X) \neq 0$.
- Q: Does there exist type n spectra for any $n \geq 0$?
- Q: Is there a systematic way to constuct type n spectra?
- S has type 0.
- Consider $S \xrightarrow{p} S \to V(1)$. Then $V(1)$ has type 1
- (Adams-Toda) When $p \geq 3$, there exists a self-map on $V(1)$ which induces $v_1 \cdot (-)$ in $K(1)$-homology. The cofiber $V(2)$ has type 2.
- (Smith-Toda) When $p \geq 5$, there exists a self-map on $V(2)$ which induces $v_2 \cdot (-)$ in $K(2)$-homology. The cofiber $V(3)$ has type 3.
- (Smith-Toda) When $p \geq 7$, $V(4)$ has type 4. (STOP HERE)

v_n-self map

Definition

Let X be a finite spectrum. A self-map $f : \Sigma^d X \to X$ is called as a v_n-map if $K(n)_*(f)$ is an isomorphism and $K(m)_*(f)$ is nilpotent if $m \neq n$.

- If X has type n, then the cofiber of a v_n-map has type $n+1$.
- If type$< n$, no v_n-maps.
  If type$> n$, the trivial map is a v_n-map.
- Any power of a v_n-map is still a v_n-map.
- Some power of f induces the multiplication of some power of v_n (up to a multiple) in $K(n)$-homology.
  $\Longleftarrow End_{K(n)_*}(K(n)_*X)$ is a finitely generated $K(n)_*$-module
- We can replace the "nilpotent" by "trivial"
  $\Longleftarrow K(m)_*(f) = 0$ when both $|v_m|$ and $|f|$ is greater than the dimension of the top cell in X.

Properties of v_n-maps

Theorem (Hopkins-Smith)

Let X, Y be type n finite spectra with v_n-maps f, g. For any map $h : X \to Y$, there exist $i, j > 0$ such that
(1) $i|f| = j|g|$ (denoted by d);
(2) The following diagram commutes up to homotopy

$$\begin{array}{ccc} \Sigma^d X & \xrightarrow{\Sigma^d h} & \Sigma^d Y \\ \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle g^j} \\ X & \xrightarrow{h} & Y \end{array}$$

- The v_n-maps are **compatible** with maps between type n spectra up to powers.
- When $h = id_X$, the v_n-maps on X are **unique** up to powers.

Proof of uniqueness

- Let $f, g \in [\Sigma^* X, X]$ be v_n-maps. Assume that $|f| = |g|$.
- Replace f, g by some powers of themselves so that $K(n)_*(f) = K(n)_*(g)$ as a multiplication of some power of v_n.
- $K(m)_*(f - g) = 0$ for all m
- Nilpotence Theorem $\Longrightarrow$ $f - g$ is nilpotent $\Longrightarrow$ $(f - g)^{p^i} = 0$ for some $i \Longrightarrow f^{p^i} = g^{p^i} + ph$
- X is p-local and finite $\Longrightarrow [\Sigma^* X, X]$ only contains p-torsion in high degrees.
- $f^{p^{i+k}} = (g^{p^i} + ph)^{p^k} = g^{p^{i+k}}$ when k is large enough.
- (**Lemma:** Let R be a ring of p-torsion. $f \in R$ such that the action $f \cdot (-) - (-) \cdot f$ on R is nilpotent. Then some power of f is in the center of R.)

Periodicity Theorem

Theorem (Hopkins-Smith)

Any finite p-local type n spectrum has a v_n-map.

- Idea of the proof:
- **Thick Subcategory Theorem:** All thick subcategories of finite p-local spectra are $\{pt\} \subset ... \subset \mathscr{F}_{n+1} \subset \mathscr{F}_n \subset ... \subset \mathscr{F}_0$, such that $\mathscr{F}_n$ consists of all spectra with type$\geq n$.
- Let $\mathbb{V}_n$ as the subcategory of all spectra admitting v_n-maps. Then $\mathscr{F}_{n+1} \subset \mathbb{V}_n \subset \mathscr{F}_n$, while we want to show $\mathbb{V}_n = \mathscr{F}_n$
- It suffices to show: **(1)** $\mathbb{V}_n$ is thick; **(2)** Some special type n spectrum admits a v_n-map.

Thickness of $\mathbb{V}_n$

- $\mathbb{V}_n$ is closed under taking cofibers:

$$\begin{array}{ccccc} \Sigma^d X & \xrightarrow{\Sigma^d h} & \Sigma^d Y & \longrightarrow & \Sigma^d C_h \\ \downarrow f & & \downarrow g & & \downarrow \\ X & \xrightarrow{h} & Y & \longrightarrow & C_h \end{array}$$

- $\mathbb{V}_n$ is closed under taking summands:
- Let f be a v_n-map on $X \vee Y$.
- Assume that f commutes with $X \vee Y \to X \to X \vee Y$.
- The composite

$$\Sigma^d X \to \Sigma^d (X \vee Y) \xrightarrow{f} X \vee Y \to X$$

becomes a v_n-map.

Construction of v_n-self maps

NOT NOW!

Spanier-Whitehead duality

Theorem

For any finite spectrum X, there is a unique finite spectrum DX (the **Spanier-Whitehead dual** of X) such that
(1) $X \mapsto DX$ is contravariant and symmetric monoidal. $DDX \simeq X$.
(2) Adjunction: $[X \wedge Y, Z] \cong [Y, DX \wedge Z]$.

- For $K(n)$-homology, we have

$$Hom_{K(n)_*}(K(n)_* X, K(n)_* Y) \cong K(n)_*(DX \wedge Y)$$

- A self-map $f \in [\Sigma^* X, X]$ corresponds to $\hat{f} \in \pi_*(DX \wedge X)$. The composition of maps is induced by the product on $DX \wedge X$:

$$DX \wedge X \wedge DX \wedge X \xrightarrow{id \wedge \epsilon \wedge id} DX \wedge S \wedge X = DX \wedge X$$

- We want some $\hat{f} \in \pi_*(DX \wedge X)$ such that $K(n)_*(\hat{f})$ is a unit element, and $K(m)_*(\hat{f}) = 0$ for $m > n$.

Adams Spectral Sequence

Theorem (Adams)

For any p-local finite spectrum R, there exists a spectral sequence

$$E_2^{s,t} = Ext_{\mathcal{A}}^{s,t}(H^*(R), \mathbb{Z}/p) \Longrightarrow \pi_{s+t}(R).$$

Here $\mathcal{A}$ is the mod p Steenrod algebra.

- The computation is EXTREMELY hard in general.
- How to make it easier?
- if $H^*(R)$ is a free $\mathcal{A}$-module?
- if $H^*(R)$ is a free module over some subalgebras of $\mathcal{A}$?
- AND replacing $\mathcal{A}$ by those subalgebras does not affect the degrees we are considering?

Structure of $\mathcal{A}$

From now on, we will assume that p is an odd prime.

Theorem (Milnor)

The dual Steenrod algebra can be expressed

$$\mathcal{A}_* = \mathbb{Z}/p[\xi_1, \xi_2, ...] \otimes E(\tau_0, \tau_1, ...)$$

with $|\xi_i| = 2p^i - 2$ and $|\tau_i| = 2p^i - 1$. The coproduct is given by

$$\Delta(\xi_n) = \sum_{0 \le i \le n} \xi_{n-i}^{p^i} \otimes \xi_i$$

$$\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{0 \le i \le n} \xi_{n-i}^{p^i} \otimes \tau_i$$

Let P_t^s, Q_i be the dual elements of $\xi_t^{p^s}$ and τ_i for any $t > s \geq 0$ and $i \geq 0$. We have $(P_t^s)^p = Q_i^2 = 0$.

Construction of v_n-self maps (again)

We want some type n spectrum X such that both $\pi_*(DX \wedge X)$ and $K(n)^*(DX \wedge X)$ are not too hard to compute:

Definition

A p-local finite spectrum X is **strongly type n** if
(a) $H^*(X)$ is a free module under $\mathbb{Z}/p[P_t^s]/(P_t^s)^p$ and under $E(Q_i)$ for any $s + t \leq n$ and $i < n$.
(b) The AHSS computing $K(n)^*X$ collapses.

Problem: conditions too strong!

Definition

A p-local finite spectrum X is **partially type n** if
(a) P_t^s and Q_i act non-trivially on $H^*(X)$ for any $s + t \leq n$ and $i < n$.
(b) The AHSS computing $K(n)^*X$ collapses.

Sketch of the construction

- **(1)** Strongly type n implies type n.
- **(2)** Existence of a v_n-map on a strongly type n spectrum.
- **(3)** A machine which transfer a partially type n spectrum to a strong one.
- **(4)** An example of a partially type n spectrum.
- (1) can be proved by studying the AHSS on $K(m)$-cohomology and the action of Q_m.
- (4): $B = B\mathbb{Z}/p$. Let B^k be its k-skeleton. Then the cofiber of $B^2 \hookrightarrow B^{2p^n}$ is partially type n.
- We will sketch the proofs of (2) and (3).

v_n-map on strongly type n spectra

- We need a connection between Adams SS and Morava K-theory:
- $k(n)$: connective Morava K-theory.
- $k(n)_* = \mathbb{Z}/p[v_n]$, can be obtained by removing all generators except v_n in BP_*.
- $H^*(k(n)) = \mathcal{A}/(Q_n)$.
- The Adams SS for $k(n)$ collapses:

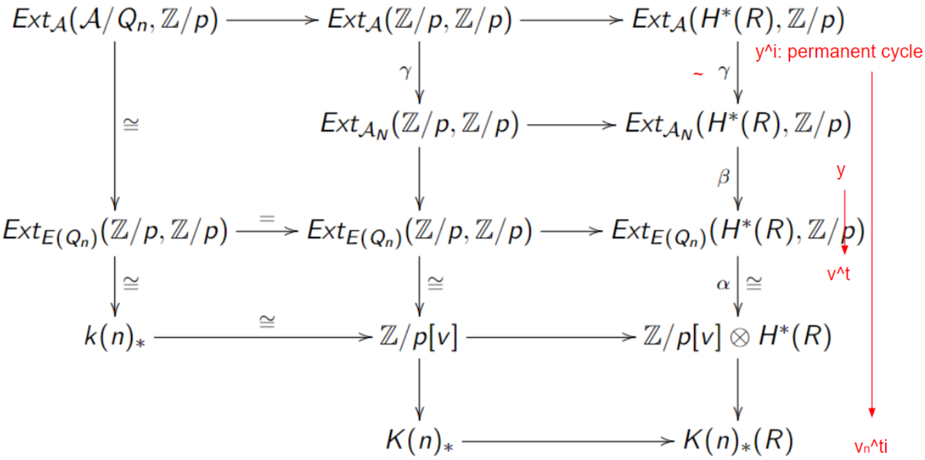
$$E_2^{*,*} = Ext_{\mathcal{A}}^{*,*}(\mathcal{A}/Q_n, \mathbb{Z}/p) \cong Ext_{E(Q_n)}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p[v_n]$$

- Assume X to be strongly type n. Let $R = DX \wedge X$. Consider the diagram:

v_n-map on strongly type n spectra

$$
\begin{array}{ccccc}
Ext_{\mathcal{A}}(\mathcal{A}/Q_n, \mathbb{Z}/p) & \longrightarrow & Ext_{\mathcal{A}}(\mathbb{Z}/p, \mathbb{Z}/p) & \longrightarrow & Ext_{\mathcal{A}}(H^*(R), \mathbb{Z}/p) \\
\downarrow \cong & & \gamma \downarrow & & \gamma \downarrow \\
 & & Ext_{\mathcal{A}_N}(\mathbb{Z}/p, \mathbb{Z}/p) & \longrightarrow & Ext_{\mathcal{A}_N}(H^*(R), \mathbb{Z}/p) \\
 & & \downarrow & & \beta \downarrow \\
Ext_{E(Q_n)}(\mathbb{Z}/p, \mathbb{Z}/p) & \xrightarrow{=} & Ext_{E(Q_n)}(\mathbb{Z}/p, \mathbb{Z}/p) & \longrightarrow & Ext_{E(Q_n)}(H^*(R), \mathbb{Z}/p) \\
\downarrow \cong & & \downarrow \cong & & \alpha \downarrow \cong \\
k(n)_* & \xrightarrow{\cong} & \mathbb{Z}/p[v] & \longrightarrow & \mathbb{Z}/p[v] \otimes H^*(R) \\
 & & \downarrow & & \downarrow \\
 & & K(n)_* & \longrightarrow & K(n)_*(R)
\end{array}
$$

v_n-map on strongly type n spectra



Transfer partial type n to strong type n

- Partial: P_s^t, Q_i act non-trivially on $H^*(X)$
  Strong: P_s^t, Q_i act freely on $H^*(X)$
- Consider the action of one fixed P_s^t or Q_i. Write $H^*(X) = F \oplus T$, where F, T are the free and "torsion" parts.
- We can assume F to be non-trivial, otherwise replace X by $X^{\wedge m}$ for large m
- $\Longleftarrow (P_s^t)^p = Q_i^2 = 0$ and Cartan formula
- Keep F and remove $T \Longleftarrow$ Smith construction

Smith construction

- There is a natural action of $\mathbb{Z}_{(p)}[\Sigma_k]$ on $X^{\wedge k}$.
- Assume that $e \in \mathbb{Z}_{(p)}[\Sigma_k]$ is idempotent. Let $eX^{\wedge k}$ be the direct limit of $X^{\wedge k} \xrightarrow{e} X^{\wedge k} \xrightarrow{e} \dots$. Define $(1-e)X^{\wedge k}$ similarly ($1-e$ is also idempotent).
- $\implies X^{\wedge k} \simeq eX^{\wedge k} \vee (1-e)X^{\wedge k}$
- On cohomology, $H^*(X)^k \cong eH^*(X)^k \oplus (1-e)H^*(X)^k$.
- Recall $H^*(X) = F \oplus T$. $H^*(eX^{\wedge k}) = eT^k \oplus F'$, where F' is free.
- It suffices to find proper k and e, such that $eT^k = 0$.

Theorem

Let V be a non-trivial $\mathbb{Z}/p$-vector space. There exists $k > 0$ and idempotent element $e_{k,V} \in \mathbb{Z}_{(p)}[\Sigma_k]$, such that for any $U \subset V$, $e_{k,V}U^k$ is non-trivial if and only if $U = V$.

(The conditions on U will be slightly different when V is graded.)